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Lifshitz holography with a probe Yang–Mills field

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ABSTRACT

Taking as a probe an $SU(2)$ gauge field with Yang–Mills action in a $3+1$ dimensional Lifshitz black hole background, we use the gauge/gravity correspondence to discuss finite temperature effects in the dual theory defined on the boundary. In order to test the dependence of results on the anisotropic scaling exponent z we consider two analytical black hole solutions with $z=2$ and $z=4$. Apart from solving the equations of motion in the bulk using a numerical approach, we also apply an analytical approximation allowing the determination of the phase transition character, the critical exponent and the critical temperature behavior as a function of z.

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Models with anisotropic scaling were introduced in condensed matter physics more than thirty years ago in order to discuss tricritical points (see [1] and references therein). They are at present actively investigated in the context of gravitational theories in which space–time anisotropic scaling improves the short distance behavior (see [2] and references therein). A link between these two issues was established by Kachru, Liu and Mulligan [3] within the framework of the gauge/gravity correspondence by searching gravity duals of non-relativistic quantum field theories. Studying the equations of motion of Einstein gravity with negative cosmological constant coupled to $p=1$ and 2-forms a solution was found in [3] with the metric taking the form

$$ds^2 = L^2\left(-r^{2z}\,dt^2 + r^2\,d\vec{x}^2 + \frac{dr^2}{r^2}\right) \tag{1}$$

where $0<r<\infty$, $d\vec{x}^2 = dx_1^2+\cdots+dx_n^2$, L is the radius of curvature of the geometry and $z \geqslant 1$. Metric (1) is invariant under anisotropic scaling of space–time coordinates

$$t \to \lambda^z t, \qquad \vec{x} \to \lambda\vec{x}, \qquad r \to \frac{r}{\lambda} \tag{2}$$

with z playing the role of the dynamical critical exponent [2]. The coordinates' inverse length dimensions are: $[t]=-z$, $[r]=+1$, $[x]=[y]=-1$. Taking Eq. (1) as a background metric, the authors in Ref. [3] extended the gauge/gravity duality to the case of models with anisotropic scaling and explored the boundary observables dual to free scalar fields in a $3+1$ dimensional bulk.

The finite temperature extension of the gauge/gravity duality requires to consider a black hole bulk metric with line element

$$ds^2 = L^2\left(-g_z(r)r^{2z}\,dt^2 + \frac{1}{g_z(r)r^2}\,dr^2 + r^2\left(dx^2+dy^2\right)\right) \tag{3}$$

where g_z vanishes at the horizon r_H. Different black hole solutions with anisotropic scaling are available [4–8] and a number of holographic studies have considered them as a background with bulk Lagrangians including different fields: charged matter, Abelian and non-Abelian gauge fields, fermions and massive Proca fields [9–14].

Using the gauge/gravity correspondence we study in the present work finite temperature effects in the dual theory defined on the boundary. We take as a probe an $SU(2)$ gauge field A_μ with Yang–Mills action, this implying that the order parameter is a vector and that one should expect a strongly anisotropic result for conductivities (among the works cited above, solely Ref. [14] has considered a vector order parameter). In order to test the dependence of results on z we shall consider two analytical $3+1$ dimensional black hole solutions with different z values: the $z=2$ black hole found in [8] and the one presented in [9] and [15] for the $z=4$ case.

The $z=2$ black hole constructed in [8] arises as a solution of the equations of motion for a $3+1$ dimensional gravitational theory with negative cosmological constant coupled to a massive vector field $\mathcal{A}_\mu$ and a scalar field Φ without kinetic term. The action reads

$$S_2 = \frac{1}{2}\int d^4x\,(R-2\Lambda) - \int d^4x\left(\frac{1}{4}\exp(-2\Phi)\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{m^2}{2}\mathcal{A}_\mu\mathcal{A}^\mu + \left(\exp(-2\Phi)-1\right)\right) \tag{4}$$

The solution of the equations of motion corresponds to a metric with line element given by Eq. (3) with

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$$g_2(r) = 1 - \frac{r_H^2}{r^2} \tag{5}$$

Starting from an action in which a Maxwell field A_μ is coupled to gravity but not directly to the massive vector field, a charged $z=4$ flat horizon black hole solution was presented in Refs. [9,15]. The action takes in this case the form

$$S_4 = \frac{1}{2}\int d^4x\,(R - 2\Lambda) - \int d^4x\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{m^2}{2}\mathcal{A}_\mu\mathcal{A}^\mu\right) \tag{6}$$

with the black holes function g_4 given by

$$g_4(r) = 1 - \frac{Q^2}{8r^4} \tag{7}$$

where Q is an integration constant related to the Maxwell field.

The black hole temperature associated with (3) is given by

$$T_z = \frac{1}{\beta} = \frac{|g_z'(r_H)|r_H^{z+1}}{4\pi} \tag{8}$$

so that for the $z=2,4$ black holes described above one has

$$T_2 = \frac{r_H^2}{2\pi}, \qquad T_4 = \frac{Q^2}{8\pi} \tag{9}$$

Note that $[T_z] = z$.

As stated above, we take as a probe an $SU(2)$ gauge field A_μ^a ($a=1,2,3$) in the black hole background (3) with $g_z(r)$ given by (5) and (7). We take from here on $L=1$. We start from the Yang–Mills action

$$S = -\frac{1}{4}\int d^4x\sqrt{|g|}F_{\mu\nu}^a F^{a\mu\nu} \tag{10}$$

The field strength $F_{\mu\nu}^a$ ($a=1,2,3$) is defined as

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \varepsilon^{abc}A_\mu^b A_\nu^c \tag{11}$$

We have taken the gauge coupling constant equal to one. The equations of motion read

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{a\mu\nu}\right) - \epsilon^{abc}F^{b\nu\mu}A_\mu^c = 0 \tag{12}$$

In order to solve these equations we shall consider the ansatz proposed in [16] for a relativistic non-Abelian gauge theory defined in an asymptotically AdS space–time

$$A = \phi(r)\tau^3\,dt + \omega(r)\tau^1\,dx \tag{13}$$

where τ^a are Pauli matrices. The gauge field inverse length dimensions are $[\phi] = z$ and $[\omega] = 1$. It will be convenient to introduce the dimensionless variable $u = r_H/r$, so that the horizon is located at $u=1$ and the asymptotic boundary at $u=0$. In terms of this variable, and inserting the black hole metric (3), Eqs. (12) reduce to

$$\phi'' + \frac{z-1}{u}\phi' - \frac{1}{r_H^2 g(u)}\phi\omega^2 = 0 \tag{14}$$

$$\omega'' + \frac{u^{z-1}}{g(u)}\partial_u\left(u^{1-z}g(u)\right)\omega' + \frac{u^{2z-2}}{r_H^{2z}g(u)^2}\omega\phi^2 = 0 \tag{15}$$

Let us discuss appropriate conditions for the gauge field components. The consistent conditions at the $u=1$ horizon are

$$\phi \sim \phi_1(1-u) + \cdots \qquad u \to 1 \tag{16}$$

$$\omega \sim \omega_H + (1-u)^2\omega_1 + \cdots \tag{17}$$

Concerning the boundary $u=0$, one has

$$\phi \sim \mu + \rho\ln(u) + \cdots \qquad z=2,\ u\to 0 \tag{18}$$

$$\omega \sim \omega_0 + \Omega u^2 + \cdots \tag{19}$$

$$\phi \sim \mu + \rho u^{2-z} + \cdots \qquad z>2,\ u\to 0 \tag{20}$$

$$\omega \sim \omega_0 + \Omega u^z + \cdots \tag{21}$$

According to the gauge/gravity correspondence μ will be identified with the chemical potential and ρ with the total charge density in the dual theory defined on the boundary.

The general solution for ϕ with $z=2$ in the normal phase takes the form

$$\phi_n = \mu_n + \rho\ln(u)$$

$$\omega = 0 \tag{22}$$

Using the horizon condition $\phi(1) = 0$, we have that

$$\mu_n = 0 \tag{23}$$

so that the chemical potential of the normal phase vanishes. In contrast, for the $z=4$ normal phase one has

$$\phi_n = \rho\left(1 - \frac{1}{u^2}\right)$$

$$\omega = 0 \tag{24}$$

and hence the chemical potential of the $z=4$ normal phase is non-vanishing, $\mu_n = \rho$.

In the $z=1$ relativistic case the divergencies of the action at the boundary are eliminated by adding counterterms. New divergent terms arise for $z \geqslant 2$ but taking a fixed charge density ρ as boundary condition makes these terms temperature independent [10]. We thus adopt this natural choice in what follows. If, as it happens in the $z=1$ case [16], ansatz (13) for a $z>1$ theory can be related to a holographic p-wave superconductor, the order parameter should then be Ω. The necessary requirement for Ω to be unsourced forces the choice of vanishing ω_0 in Eq. (21) or Eq. (24). The divergencies of the action in the normal $\omega = 0$ phase and the superconducting $\omega \neq 0$ one coincide leading to a finite free energy difference, as we shall see below.

We shall now proceed to calculate the free energy $\mathcal{F}$, related to the Euclidean on-shell action according to

$$\mathcal{F} = T_z S_E|_{on\,shell} \tag{25}$$

Before proceeding to the Wick rotation of the action we insert the ansatz (13) in Eq. (10)

$$S = -\frac{V}{2T_z}\int du\,\frac{1}{u^{3+z}}\left(-r_H^{2-z}u^{2z+2}(\phi')^2 - r_H^{-z}\frac{u^{2z+2}}{g_z(u)}\omega^2\phi^2 + r_H^z u^4(\omega')^2 g_z(u)\right) \tag{26}$$

where V is the two dimensional boundary spatial volume.

We start with the $z=2$ case. Integrating by parts Eq. (26) and using the equations of motion we get

$$\frac{T_2}{V}S = \frac{1}{2}\left[\left(u\phi\phi'\right)\Big|_{u=\epsilon} - \frac{r_H^2 g(u)}{u}\omega'\omega\Big|_{u=\epsilon}\right] - \frac{1}{2}\int du\,\frac{u}{r_H^2 g(u)}\phi^2\omega^2 \tag{27}$$

Here ϵ is a cut-off which will be put to zero at the end of the calculations. As discussed above, we choose to work in the canonical ensemble and hence we add a boundary term to the action [17]

$$-\frac{1}{2}\int dt\, d^2x \sqrt{-g} A_\mu F^{u\mu}\big|_{u=\epsilon}$$
$$= \frac{V}{2T_2}\left[\left(u\phi\phi'\right)\big|_{u=\epsilon} - \frac{r_H^2 g(u)}{u}\omega'\omega\bigg|_{u=\epsilon}\right] \tag{28}$$

After a Wick rotation, using Eq. (25) and the boundary behavior of the gauge field the free energy density at fixed charge takes the form

$$\frac{\mathcal{F}}{V} = -\rho\mu + \frac{1}{2}\int du \frac{u}{r_H^2 g(u)}\phi^2\omega^2 + \frac{1}{2}\rho^2 \ln(u)\big|_{u=\epsilon} \tag{29}$$

The logarithmic divergent term in the r.h.s. will play no role when comparing the free energies of the solutions with $\omega \neq 0$ with that of the normal $\omega = 0$ case which has the same divergent term so that one ends with

$$\frac{\Delta\mathcal{F}}{V} = \frac{\mathcal{F}-\mathcal{F}_n}{V} = -\rho\mu + \frac{1}{2}\int du \frac{u}{r_H^2 g(u)}\phi^2\omega^2, \quad z=2 \tag{30}$$

Proceeding in the same way in the $z = 4$ case, we find

$$\frac{\Delta\mathcal{F}}{V} = -\rho(\mu-\mu_n) + \frac{1}{2}\int du \frac{u^3}{r_H^2 g(u)}\phi^2\omega^2, \quad z=4 \tag{31}$$

where μ_n is the chemical potential of the normal phase.

Before discussing the numerical solutions of Eqs. (14)–(15) we shall develop an analytic approach which allows to calculate the critical temperature and the behavior of the order parameter with remarkable accuracy. The method is based on a proposal presented in Ref. [18] which consists in obtaining solutions in close form by imposing conditions of continuity and smoothness at a point u_m intermediate between the boundary ($u = 0$) and the horizon ($u = 1$). Originally u_m was arbitrarily chosen to be 1/2 and rather good results in comparison with more involved numerical methods were obtained. As discussed in [19] the agreement stems from rather elementary considerations on perturbation of Schrödinger-like equations. We here extend the method in order to determine u_m from a simple free energy argument and in this way, the method turns out to be a powerful tool to study the behavior of the system as a function of z.

In practice, we shall consider expansions of the fields near $u = 1$ and $u = 0$ and determine their leading orders coefficients by connecting the expansions at $u = u_m$. We start from the case $z = 2$. For the solution near the horizon ($u = 1$) we have, up to second order in the expansions of the fields we call $\omega^h(u)$ and $\phi^h(u)$,

$$\omega^h(u) = \omega_0^h + \omega_1^h(u-1) + \frac{1}{2}\omega_2^h(u-1)^2$$
$$\phi^h(u) = \phi_0^h + \phi_1^h(u-1) + \frac{1}{2}\phi_2^h(u-1)^2 \tag{32}$$

where ω_i^h, ϕ_i^h, are constants to be determined. The superscript h indicates that the expansion is performed near the horizon. Now, conditions (16) at $u = 1$ imply that

$$\phi_0^h = 0, \qquad \phi_2^h = \frac{1}{4}\phi_1\left(2 + \frac{\omega_H^2}{r_H^2}\right) \tag{33}$$

$$\omega_1^h = 0, \qquad \omega_2^h = -\frac{1}{16}\frac{\phi_1^2\omega_H}{r_H^4} \tag{34}$$

We now insert these relations in Eq. (32) and match the expansions of ω and ϕ and their derivatives at $u = u_m$. From this we get

$$\phi_1 = -\frac{4r_H^2}{\sqrt{1-u_m}}, \qquad \Omega = \frac{\omega_H}{u_m} \tag{35}$$

$$\omega_H = \left(2r_H^2\frac{2-u_m}{1-u_m} - \frac{\rho}{2u_m(1-u_m)^{1/2}}\right)^{1/2} \tag{36}$$

At this point we can write r_H in terms of the temperature T using Eqs. (9)

$$\Omega = \frac{1}{u_m}\left(4\pi T_2\frac{2-u_m}{1-u_m} - \frac{\rho}{2u_m(1-u_m)^{1/2}}\right)^{1/2} \tag{37}$$

Determination of the point at which the order parameter Ω vanishes leads to the critical temperature

$$T_2^c = \frac{1}{8\pi}\frac{(1-u_m)^{1/2}}{u_m(2-u_m)}\rho. \tag{38}$$

One can also infer the temperature dependence of the condensate close to the phase transition

$$\Omega = \mathcal{N}_2(u_m)\left(4\pi T_2^c\right)^{1/2}\left(1-\frac{T}{T_2^c}\right)^{1/2} \tag{39}$$

$$\mathcal{N}_2(u_m) = \frac{1}{u_m}\left(\frac{2-u_m}{1-u_m}\right)^{1/2} \tag{40}$$

Similar calculations with $z = 4$ yield, using $Q^2 = 8r_H^4$

$$\phi_1 = -\frac{Q^2}{(1-u_m/2)^{1/2}(1-u_m)^{1/2}}, \qquad \Omega = \frac{\omega_H}{u_m^3(2-u_m)} \tag{41}$$

$$\omega_H = -2^{1/4}\left(Q\frac{4-3u_m}{u_m-1} + \frac{4\rho}{Q^2}\frac{(2-u_m)^{1/2}}{u_m^3(1-u_m)^{1/2}}\right)^{1/2} \tag{42}$$

which defines the critical temperature as

$$T_4^c = \frac{(2-u_m)^{1/2}(1-u_m)^{1/2}}{2^{5/2}\pi u_m^3(4-3u_m)}\rho \tag{43}$$

Finally for the behavior of the order parameter near the critical temperature we obtain

$$\Omega = \mathcal{N}_4(u_m)\left(16\pi T_4^c\right)^{1/4}\left(1-\frac{T}{T_4^c}\right)^{1/2} \tag{44}$$

$$\mathcal{N}_4(u_m) = \frac{1}{u_m^3}\frac{(4-3u_m)^{1/2}}{(2-u_m)(1-u_m)^{1/2}} \tag{45}$$

We then see that both for $z = 2$ and $z = 4$ the behavior of Ω near the critical point reveals a typical scenario of a second order phase transition, with an ordered phase $\omega \neq 0$ for $T < T_z^c$ in agreement with the results in the most diverse relativistic models explored using the gauge/gravity duality, with critical exponents coinciding with those obtained within the mean field approximation, independently of the choice of u_m.

To confirm the results obtained above we have still to compare the free energy associated to the solution we have found with that for the disordered (normal) phase which corresponds to $\omega = 0$. If the difference of free energies $\Delta\mathcal{F}$ is negative below the critical temperature then a phase with non-vanishing order parameter will be preferred for $T < T_z^c$. This fact will allow us to determine u_m as a function of ρ, from minimization of $\Delta\mathcal{F}$ written in terms of expansion (32) from the horizon to u_m and of expansions (19)–(20) from the boundary to u_m,

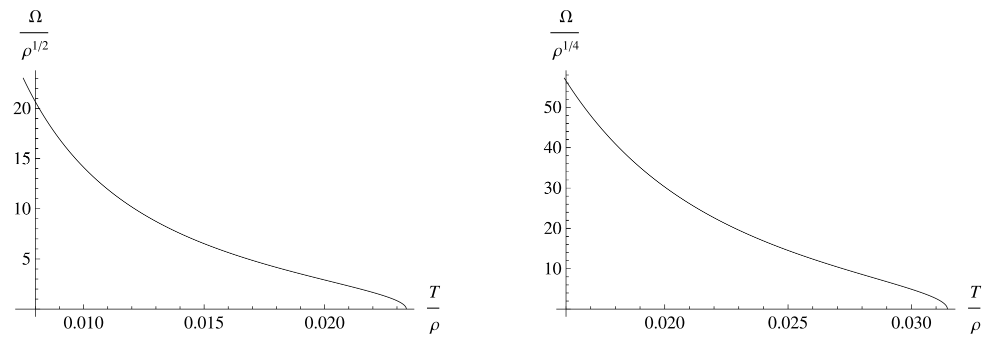

Fig. 1. The numerical result for the condensate as a function of temperature for the $z=2$ (left) and $z=4$ (right) cases. The condensate goes to zero as $(T-T^c)^{1/2}$ in both cases thus confirming the analytic results (Eqs. (39), (44)).

$$\frac{\Delta\mathcal{F}}{V} = -\rho(\mu-\mu_n) + \frac{1}{2}\int_0^{u_m} du\,\frac{u}{r_H^2 g(u)}\left(\phi^b\omega^b\right)^2 + \frac{1}{2}\int_{u_m}^{1} du\,\frac{u}{r_H^2 g(u)}\left(\phi^h\omega^h\right)^2 \tag{46}$$

where ϕ^b and ω^b are given by (19) for $z=2$ and (21) for $z=4$. Note that we have not included the divergent term in (46) since we are working at fixed ρ and hence such term is u_m independent. Minimization of Eq. (46) gives a solution for u_m which, inserted in Eqs. (38) and (43) gives the following critical temperature coefficients

$$T_2^c = 0.022\rho, \qquad T_4^c = 0.025\rho \tag{47}$$

We will confirm below this scenario and compare these results with those obtained by solving the equations of motion numerically. Before doing this let us note that the critical temperature obtained analytically increases when changing from the $z=2$ to the $z=4$ system. To determine whether this is a general behavior for arbitrary values of z is relevant in connection with the theory of Fermi liquids [10]. To analyze this issue in more general terms one can take for illustrative purposes the following black hole function

$$g(u;z) = 1-u^z \tag{48}$$

which includes the actual Lifshitz $z=2$ and $z=4$ black hole solutions studied previously. From Eq. (8) one can write r_H in terms of z and T and then, using the analytical approach one can confirm that, for black holes of the form (48), T_z^c is a growing function of z for $z\geqslant 2$ for any choice of u_m.

We now proceed to solve the equations of motion numerically. The strategy is the following: the solutions are searched as functions of the parameters ω_H and ϕ_1 at the horizon with vanishing constant term for ϕ and with general non-vanishing ω_0 at the boundary (see Eqs. (16)–(19)). Then the numerical system is solved searching possible values of ϕ_1 at the horizon for which ω_0 vanishes. In this way we have obtained a set of solutions for different field values at the horizon. The existence of several solutions satisfying the appropriate boundary conditions, each one corresponding to a different value of ϕ_1, is a phenomenon already present in the relativistic case [16]. For increasing values of ϕ_1 the solution for ω has an increasing number of nodes n. Now, evaluation of the free energy shows that it increases with the number of nodes and hence we conclude that solutions with $n\geqslant 1$ are energetically disfavored so that we shall solely discuss the zero-node solution.

Our numerical solution confirms the results found analytically: a finite temperature continuous symmetry breaking phase transition takes place both for $z=2$ and $z=4$. As shown in Fig. 1 the system condensates at a critical temperature T_c. The behavior near T_c can be seen, by fitting the curve, to correspond to a second order transition with critical exponent $1/2$ as advanced by the analytical result, Eqs. (39)–(44). It should be stressed that profiles for $z=2$ and $z=4$ are strikingly resemblant. What distinguishes the two cases is the value of the critical temperatures:

$$T_2^c = 0.023\rho, \qquad T_4^c = 0.031\rho \tag{49}$$

Comparing these values with those obtained previously using the analytic approach Eqs. (47) we find a remarkable agreement.

Note that at low temperature the condensates appear to diverge as a negative power of the temperature. This behavior was already encountered in the relativistic $z=1$ case, both for s-wave [20] and p-wave [16] holographic superconductors and can be ascribed to the relevance of back-reaction when the condensate becomes too large so that the probe approximation is no more valid. Using again Eq. (48) as an illustration, our analytical approach shows that the behavior of the condensate for T small – in the range of validity of the probe approximation – is $\Omega \propto T^{-(z-2)/2z}$ for $z\geqslant 2$ independently of the choice of the matching point u_m.

Using formulae (30)–(31) we have computed numerically the free energy difference between the ordered and disordered phases (see Fig. 2) confirming that, both for $z=2$ and $z=4$, the ordered phase is preferred below the critical temperature T_z^c whose values coincide with those given by (49).

Finally, we shall compute the electromagnetic response to small time dependent perturbations of the Yang–Mills field in the ordered phase. To do this, we start from the gauge field ansatz (13) (that we shall denote $A_\mu^{ord}(u)$ for clarity) and following [16] we consider the perturbation

$$A_\mu = A_\mu^{ord}(u) + a_\mu(u,t) \tag{50}$$

$$a_\mu\,dx^\mu = e^{-iw_f t}\left[\left(a_t^1\tau^1 + a_t^2\tau^2\right)dt + a_x^3\tau^3\,dx + a_y^3\tau^3\,dy\right] \tag{51}$$

with w_f the frequency associated to the perturbation. The linearized Yang–Mills equations read

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\mathcal{F}^{\mu\nu a}\right) - \epsilon^{abc}\mathcal{F}_b^{\nu\mu}A_{\mu c} - \epsilon^{abc}F_b^{\nu\mu}a_{\mu c} = 0 \tag{52}$$

where

$$\mathcal{F}_{\mu\nu}^a = \partial_\mu a_\nu^a - \partial_\nu a_\mu^a - \epsilon^{abc}A_{\mu b}a_{\nu c} + \epsilon^{abc}A_{\nu b}a_{\mu c} \tag{53}$$

Using Eqs. (13), (51) one finds four second order equations

$$a_y^{3\,\prime\prime} + \left(\frac{1-z}{u} + \frac{g'(u)}{g(u)}\right)a_y^{3\,\prime} + \frac{w_f^2 u^{2z-2}}{r_H^{2z}g^2(u)}a_y^3 - \frac{\omega^2}{r_H^2 g(u)}a_y^3 = 0 \tag{54}$$

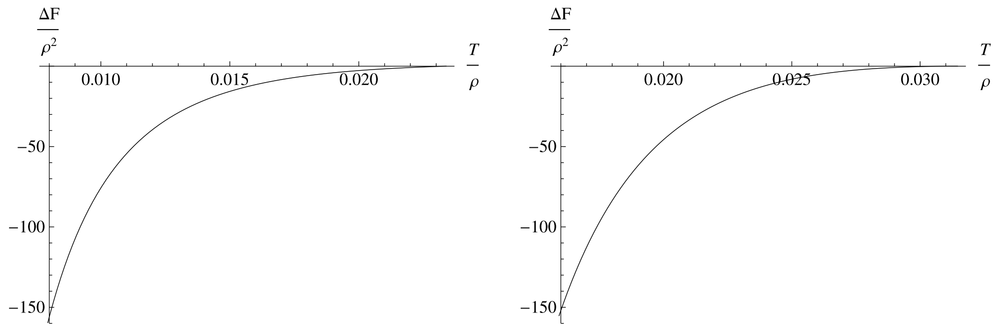

Fig. 2. The free energy difference between the condensed and the uncondensed phase as a function of temperature for the $z = 2$ (left) and the $z = 4$ (right) models.

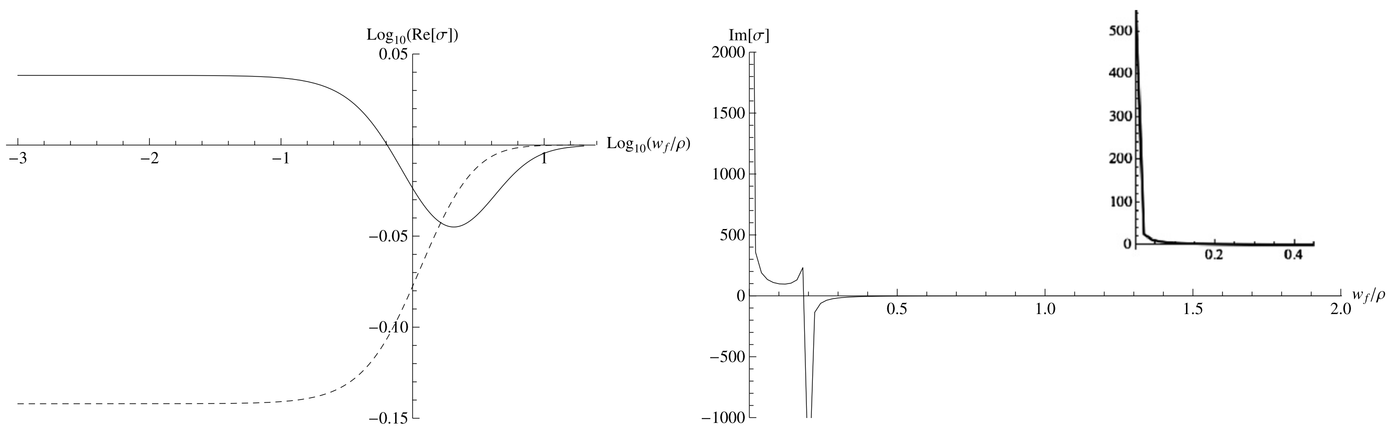

Fig. 3. Real and imaginary parts of conductivity as a function of the frequency for $T/\rho = 0.022$ for the $z = 2$ system. The solid line corresponds to σ_{xx} and the dashed one to σ_{yy}. The insert figure on the right displays a detail of the imaginary part of σ_{yy} rendering visible the pole at $\omega_f = 0$.

$$a_x^{3\prime\prime} + \left(\frac{1-z}{u} + \frac{g'(u)}{g(u)}\right) a_x^{3\prime} + \frac{u^{2z-2}}{r_H^{2z} g^2(u)}\left(-i w_f \omega a_t^2 + w_f^2 a_x^3 - \omega \phi a_t^1\right) = 0 \tag{55}$$

$$a_t^{1\prime\prime} + \frac{z-1}{u} a_t^{1\prime} + \frac{\omega \phi}{r_H^2 g(u)} a_x^3 = 0 \tag{56}$$

$$a_t^{2\prime\prime} + \frac{z-1}{u} a_t^{2\prime} - \frac{\omega^2}{r_H^2 g(u)} a_t^2 - \frac{i w_f \omega}{r_H^2 g(u)} a_x^3 = 0 \tag{57}$$

and two first order equations

$$i w_f a_t^{1\prime} + \phi a_t^{2\prime} - \phi a_t^{2\prime} = 0 \tag{58}$$

$$i w_f a_t^{2\prime} - \phi a_t^{1\prime} + \phi' a_t^1 - g(u) u^{2-2z}\left(\omega \partial_u - \omega'\right) a_x^3 = 0 \tag{59}$$

Let us concentrate on the case $z = 2$. The choice of the electromagnetic perturbation should correspond to a wave traveling away from the conformal boundary at $u = 0$ (an "in-going" wave). In the present case one has, near the horizon

$$a_y^3 = \alpha\left(1-u^2\right)^{-\frac{i w_f}{2 r_H^2}}(1 + \cdots)$$

$$a_x^3 = \beta(1-u)^{-i\frac{w_f}{2 r_H^2}}\left(1 + a_1(1-u) + \cdots\right) \tag{60}$$

$$a_t^1 = \gamma(1-u)^{-i\frac{w_f}{2 r_H^2}}\left(a_2(1-u)^2 + \cdots\right)$$

$$a_t^2 = \delta(1-u)^{-i\frac{w_f}{2 r_H^2}}\left(a_4(1-u) + \cdots\right) \tag{61}$$

with $\alpha, \ldots, \gamma$ dimensionful constants. At the boundary we have instead

$$a_y^3 = a_{y(0)}^3 + u^2 a_{y(2)}^3 + \cdots, \qquad a_x^3 = a_{x(0)}^3 + u^2 a_{x(1)}^3 + \cdots \tag{62}$$

$$a_t^1 = a_{t(0)}^1 + a_{t(1)}^3 \ln(u) + \cdots, \qquad a_t^2 = a_{t(0)}^2 + a_{t(1)}^2 \ln(u) + \cdots \tag{63}$$

where all coefficients a_i can be determined as functions of ω and ϕ at the horizon once w_f is specified.

The conductivity can then be obtained using Ohm's law. Following [16] for the case of non-Abelian gauge fields, the conductivity components are

$$\sigma_{yy} = -i\frac{r_H^2 a_{y(2)}^3}{w_f a_{y(0)}^3}$$

$$\sigma_{xx} = -\frac{i r_H^2}{w_f a_{x(0)}^3}\left(a_{x(1)}^3 + \Omega\frac{i w_f a_{t(0)}^2 + \mu a_{t(0)}^1}{\mu^2 - w_f^2}\right) \tag{64}$$

We show the numerical solution for the real and imaginary parts of σ_{xx} and σ_{yy} for the $z = 2$ system in Fig. 3. As in the relativistic case the conductivity components approach 1 at large w_f. We observe the formation of a gap in the real part of σ_{yy} as it happens in the case of a Maxwell field coupled to a scalar [20] and in the purely Yang–Mills [16] bulk Lagrangians cases. There is a pole in the imaginary parts of σ_{xx} and σ_{yy} at $w_f = 0$ characteristic of superconducting behavior. There is a second pole in the imaginary σ_{xx} at $w_f = w_f^* = 0.199\rho$ at $T/\rho = 0.022$ accompanied by the corresponding delta function in its real part, in agreement with Kramers–Kronig relations (this delta function is not represented in Fig. 3 left since the numerical procedure can only render continuous functions). The w_f^* value obtained numerically satisfies $w_f^* = \mu$ as expected from Eq. (64). This pole is absent in the analysis of [14] for a bulk Yang–Mills Lagrangian in the background of a different $z = 2$ Lifshitz black hole (the one presented in [4] with $g_2(u) = (1 - u^4)$ arising in the case in which the dilaton field is dynamical, instead of the one we have used, Eq. (5)). In [14] such absence was attributed to the logarithmic behavior of A_0 resulting from the $z = 2$ scaling. Our result shows that for the $z = 2$ black hole background that we used such logarithmic behavior does not prevent the existence of this pole.

The analysis of the $z = 4$ theory follows similarly and the behavior of conductivity components is qualitatively the same. We also find in this $z = 4$ case, with $g_4(u) = 1 - u^4$, a second pole located at $w_f^* = 20.5\rho$ for $T/\rho = 0.022$.

We shall end this work with a brief summary and a discussion of our results. We have studied finite temperature effects in two models with different dynamical critical exponent using the gauge/gravity correspondence. Looking for a vector order parameter and inspired by Gubser and Pufu's work on $z = 1$ p-wave holographic superconductors [16], we have chosen as gravity dual a Yang–Mills theory in the gravitational background of Lifshitz black holes with $z = 2$ and $z = 4$. Apart from solving the equations of motion in the bulk using a numerical approach, we have also extended the analytical approximation developed in [18,19] which allows to reproduce the numerical results with remarkable simplicity and precision.

Although one could presume that the anisotropic scaling of the background metric would lead to a critical behavior differing from the one found in [16] for $z = 1$, our results show instead a remarkable resemblance with the relativistic case. In particular, the condensate has the typical $(T_z^c - T)^{1/2}$ mean field behavior for T close to the critical temperature T_z^c both for $z = 2$ and $z = 4$. The dependence on z only affects the coefficient in the critical temperature which grows with z, a behavior that could be argued to be valid for arbitrary z, as we illustrated applying our analytic approach to a heuristic black hole function $g(u; z)$ defined in Eq. (48). Using the same approach we were able to extract the condensate behavior in the range of small temperatures where the probe approximation is valid, finding that $\Omega \propto T^{-(z-2)/2z}$, in total agreement with the numerical calculations. All these results confirm that the analytic approximation developed in [18,19] and refined here has proved to be sufficiently accurate as to avoid the necessity to resort to numerical methods.

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