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Use of a non-relativistic basis for describing the low energy meson spectrum

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Abstract. We justify the use of harmonic oscillator eigenfunctions for expanding the fermionic fields of an effective field theory and evaluate its utility for solving a Hamiltonian inspired by the QCD formalism in the Coulomb Gauge. Since the functions involved in such expansion are non-relativistic, the Talmi-Moshinsky transformations can be used to recover the translational invariance of the center of mass of the mesonic states. Finally, many-body methods and an *a posteriori* flavor mixing procedure are used to compute a preliminary spectrum for the mesons below 1 GeV.

1. Introduction

In the low energy regime (below 1 GeV) the strong interaction coupling constant α_s is of the order of the unit and therefore it is impossible to model the color interaction as a perturbation. Despite the fact that there has been a great effort in describing the non-perturbative regime of QCD (NP-QCD), there is still not a satisfactory solution which fully describes this regime.

There are many phenomenological models that have tried to formulate an effective NP-QCD theory. One of the first to do this was the MIT bag model [1], whose main disadvantage is to be unrealistic in addition to requiring external parameters to the theory.

So far, mainly the Lattice QCD [2] is able to derive non-perturbative results from first principles, for example, it has been able to describe the hadronic states at low energy [3, 4]; as well as to the highly excited mesonic states [5, 6]. However, this approach is intrinsically limited by the rapid rise in computational effort required as one approaches the physically relevant limit of small lattice spacing and large lattice volume. In addition, it has the less important disadvantage that recovering the rotation symmetry is unnecessarily complicated due to the lattice discretization [7].

Another way to approach the NP-QCD calculations is via solving the Dyson-Schwinger equations (DSEs). The main advantage of such formalism is that it represents a continuum method and thus, both the UV and the IR regions can be investigated. In particular, DSEs are also able to describe the dynamical mass generation of the Yang-Mills sector of the theory in a self-consistent way [8, 9, 10]. Nevertheless, the DSEs form an infinite set of coupled integral

equations. Therefore the system has to be truncated and cannot be solved completely and there are no clear prescriptions on how to perform such a truncation.

For these reasons, the overall objective is to formulate a model that represents an alternative to perform non-perturbative calculations based on an effective field theory. The present model is meant to offer the following properties:

- That requires less computing power than lattice calculations. This includes the fact that the codes involved are easier to program and can even be run by an standard-performance personal computer.
- That requires the least possible number of free parameters. This is achieved through the use of variational methods to relate the parameters to each other, in such manner that not all of them are free but arise naturally as a function of the theory to be evaluated.
- That offers a more intuitive physical interpretation than the approaches mentioned above. This is expected to be achieved after an *a posteriori* verification of the effectiveness of the assumed ansatz.
- That is simple yet realistic enough, *i.e.*, by simulating the most important features of QCD, it can be expected to represent a good approximation to reality.
- Last but not least, it is expected that the numerical solutions generate a sufficiently rich spectrum to compare with the results of other non-perturbative methods and even with the experimental mesonic spectrum.

2. Characteristics of the model

As mentioned above, one of the goals of the model is to be able to reproduce the important aspects of the theory by combining the following properties:

- One considers an effective QCD Hamiltonian where the quarks interact via the non-relativistic Cornell Potential, which simulates the effect of the gluons [11].
- The main characteristic of the model is that the spatial distribution of the fermionic fields is expanded in the non-relativistic 3D-harmonic oscillator (3D-h.o.) basis in the coordinate space [12, 13], *i.e.*, the angular functions are the spherical harmonics while the radial part is given by

$$\begin{aligned} R_{nl}(r) &= \mathcal{N}_{nl} r^l e^{-\frac{\lambda}{2}r^2} L_n^{l+\frac{1}{2}}(\lambda r^2) \\ \mathcal{N}_{nl} &= \left[\frac{2(n!)}{\Gamma(n+l+\frac{3}{2})}\right]^{\frac{1}{2}} \lambda^{\frac{3}{4}+\frac{l}{2}} \\ n &= \tfrac{N-l}{2} \ , \end{aligned} \tag{1}$$

  where $L_n^{\alpha}(x)$ are the associated Laguerre polynomials. This procedure will bring several advantages that are discused in Subsection 2.1.
- Even though this basis is not Lorentz invariant, it is possible to recover the Galilei invariance of the center of mass (CM) with the help of the so called Talmi-Moshinsky (TM) transformations [14, 15].
- Many-body methods such as the Tamm-Dankoff method (TDA) and the Random Phase Approximation (RPA) are used to diagonalize the effective Hamiltonian [16].
- Flavor symmetry breaking is set by introducing an *a posteriori* flavor mixing procedure [17].

2.1. The election of a basis

We are not the first group to have implemented the many-body methods for finding the mesonic spectrum related to an effective static potential [18, 19]. Nevertheless, the fundamental difference proposed by the present model is the use of the 3D-h.o. basis for expanding the fermionic fields in the position space.

In principle any complete basis can be used to construct the Fock space, however, the structure of the vacuum will depend on the chosen basis: The better the ansatz is to represent the vacuum of the interacting theory, the more efficiently it will describe the physics involved.

The fields (fermionic or bosonic) are usually expanded in the plane waves basis, which are solutions to the free propagation problem, that is, to the Dirac or Klein-Gordon equation respectively. The Quantum Field Theories have been deeply explored in such formalism that comes very useful when perturbative expansions are made, but little has been done to give formalism to a field theory expanded in 3D-h.o. solutions.

The election of such an initial basis for expanding the fields is due to the following reasons:

- As mentioned in Section 1, lattice QCD calculations are limited by the finite spacing of the grid. Thus, the expansion of the fields in 3D-h.o. eigenfunctions can be interpreted as a discrete mapping of continuous solutions that may avoid the loss of information in the small spacing limit.
- Due to confinement, the fermionic fields are expected to be restricted to a finite volume, forming *localized* hadrons. This is why the oscillator basis is intuitively more adequate than plane waves, since it establishes a characteristic lenght for the fields in the position space: By expanding the fields in such basis, the color charge distribution functions are suppressed over long distances by an exponential $e^{-\frac{\lambda}{2}r^2}$, while the function of the hadron's CM is suppressed by a weight function $e^{-\frac{\lambda}{2}R^2}$.
- Although the 3D-h.o. basis is not Lorentz invariant, it is possible to recover the Galilean invariance of the mesons' CM by using the TM transformations [14]. This means that the individual quarks will not be invariant under spatial translations, however, the important degrees of freedom to be considered in QCD at low energies are light mesons and not their constituent components.
- It is not necessary to introduce any sort of cut-off in the integrals involved. This is because the integrals in the position space are finite for both short and large relative distances. However, a cut-off has to be introduced numerically by truncating the expansion on the functions up to a certain number of oscillator quanta N_{cut}.
- After implementing the transformation to the CM frame, the integrals involved in the matrix elements of the interaction are decoupled, so that they can be expressed in a single one-dimensional integral, which will result to be a *Talmi integral* with analytical results.
- Free-propagation solutions may not be the best approach to a strong interacting theory, while in the CM frame the 3D-h.o. basis implies a significant simplification of the problem as well as fast convergence of the solutions [15].
- This basis seems to be suitable in order to extend the analysis in many directions, like adding gluonic degrees of freedom or evaluating other hadronic states.

3. Construction of the effective Hamiltonian

In the same way as in QED, fixing the Coulomb gauge gives rise to a local and instantaneous potential. Lattice calculations have shown that this potential is approximately of the form $-\frac{\alpha}{r}+\beta r$ [20, 21], where $r=|\boldsymbol{x}-\boldsymbol{y}|$ is the relative distance between two points of the color charge densities. Such interaction is known in the literature as Cornell's potential and its linear component reproduces the phenomenon of confinement, since it makes energetically less favorable

to increase the distance between charges. Because of this, the Cornell potential simulates the effect of gluons, which makes possible the treatment of a simplified theory where the only fields involved are fermionic.

This approach to the problem is then equivalent to replacing the Faddeev-Popov operator by the static potential

$$\langle a,\boldsymbol{x}|\frac{1}{\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{D}}}(-\boldsymbol{\nabla}^2)\frac{1}{\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{D}}}|a'\boldsymbol{y}\rangle \longrightarrow V(|\boldsymbol{x}-\boldsymbol{y}|)=\frac{-\alpha}{|\boldsymbol{x}-\boldsymbol{y}|}+\beta|\boldsymbol{x}-\boldsymbol{y}| \ . \tag{2}$$

Thus, the effective Hamiltonian to be solved in the present model is

$$\begin{aligned} H &= \boldsymbol{H}_{K_q}+\boldsymbol{H}_{m_q}+\boldsymbol{H}^{q-q}_{Coulomb} \\ \boldsymbol{H}_{K_q} &= \int \boldsymbol{\psi}^\dagger(\boldsymbol{x})[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}]\boldsymbol{\psi}(\boldsymbol{x})d\boldsymbol{x} \\ \boldsymbol{H}_{m_q} &= \int \boldsymbol{\psi}^\dagger(\boldsymbol{x})[\beta m_0]\boldsymbol{\psi}(\boldsymbol{x})d\boldsymbol{x} \\ \boldsymbol{H}^{q-q}_{Coulomb} &= -\tfrac{1}{2}g^2\delta_{a'a}\int \rho^{(q)}_a(\boldsymbol{x})V(|\boldsymbol{x}-\boldsymbol{y}|)\rho^{(q)a}(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y} \ , \end{aligned} \tag{3}$$

where $\alpha_i=\gamma^0\gamma^i$ and $\beta=\gamma^0$ are related to the Dirac matrices γ^μ; while m_0 is the bare mass of the quark. On the other hand,

$$\rho^{(q)a}=\sum_{c'c}\boldsymbol{\psi}^\dagger{}_{cf}T^a_{cc'}\boldsymbol{\psi}_{c'f} \ , \tag{4}$$

is the color charge density *restricted* to the quark sector (q) only.

3.1. The free propagation Hamiltonian

The free propagation Hamiltonian is conformed by the kinetic and mass terms restricted to the quark sector only.

$$\boldsymbol{K} \ = \ \boldsymbol{H}_{K_q}+\boldsymbol{H}_{m_q}=\int \boldsymbol{\psi}^\dagger(\boldsymbol{x})[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+\beta m_0]\boldsymbol{\psi}(\boldsymbol{x})d\boldsymbol{x} \ ,$$

where, in the current status of the model, the bare mass for the u and d quarks is assumed to be degenerated though unequal to the mass of the quark s, i.e.,

$$(m_0)_{YT}=m_{u,d}\delta_{Y,\frac{1}{3}}\delta_{T,\frac{1}{2}}+m_s\delta_{Y,-\frac{2}{3}}\delta_{T,0} \ . \tag{5}$$

The operator $\boldsymbol{K}$ is not diagonal in the 3D-h.o. basis, and thus it is necessary to define a unitary transformation that diagonalizes it.

$$Q^\dagger_{\tau(Nl)jmcf} \ = \ \sum_{\pi k\lambda}\left(\alpha^j_{\tau(Nl)\lambda\pi k}\right)^* q^\dagger_{\lambda\pi kjmcf}\delta_{\pi,(-1)^{\frac{1}{2}-\tau+l}} \ . \tag{6}$$

This will also allow to identify the particle-hole operators $q^\dagger_{\pm\frac{1}{2}}$ with the quark(antiquark) operators $b^\dagger(d)$ via the definitions

$$\begin{aligned} b^\dagger_{\pi kjmcf} &\equiv q^\dagger_{\frac{1}{2}\pi kjmcf} \\ d_{\pi kjmcf} &\equiv q^\dagger_{-\frac{1}{2}\pi kjmcf} \ . \end{aligned} \tag{7}$$

This way, after performing the trasformation of eq. (6), the free propagation Hamiltonian can be written as a one-body operator of the form

$$\boldsymbol{K} \ = \ \sum_{kj}\epsilon_{kjYT}\left(\boldsymbol{b}^\dagger_{kjYT}\cdot\boldsymbol{b}^{kjYT}-\boldsymbol{d}_{kjYT}\cdot\boldsymbol{d}^{\dagger kjYT}\right) \ ,$$

where ϵ_{kjYT} are the eigenenergies obtained by diagonalizating the $\boldsymbol{K}$ in the harmonic oscillator basis.

At this point it is better to make an observation: as the free propagation Hamiltonian is, by construction, diagonal in the basis of the operators b's and d's (denoted as $(\lambda\pi k)$-basis); and knowing that the exact solution for said Hamiltonian are the plane waves, it is to conclude that both basis are closely related.

In fact, due to the completeness of the 3D-h.o. basis, the solutions shall tend to reproduce plane waves in the limit where all the quanta of oscillation are taken into account $N_{cut} \to \infty$. This is the reason why the $(\lambda\pi k)$-basis represents the connection that translates the excitations described by the operators b's and d's with the asymptotic states of a more standard field theory, where said excitations can actually be interpreted as physical quarks and antiquarks respectively.

3.2. The interaction Hamiltonian

Evidently, a one-body operator has a trivial CM, nevertheless, for the interaction Hamiltonian in (3) one obtains a two-body operator whose CM should in principle be removed in order to recover Galilean invariance. For doing so, the so called TM transformations are introduced:

$$[\Psi_{n_a l_a}(\boldsymbol{x}) \otimes \Psi_{n_b l_b}(\boldsymbol{y})]^L_M = \sum_{n_r l_r n_R l_R} \langle n_r l_r, n_R l_R; L | n_a l_a n_b l_b; L\rangle \, [\Psi_{n_r l_r}(\boldsymbol{r}) \otimes \Psi_{n_R l_R}(\boldsymbol{R})]^L_M \ , \tag{8}$$

where the Jacobi coordinates $\boldsymbol{R} = \frac{1}{\sqrt{2}}(\boldsymbol{x}+\boldsymbol{y})$ and $\boldsymbol{r} = \frac{1}{\sqrt{2}}(\boldsymbol{x}-\boldsymbol{y})$ refer to the CM and relative motions of two color charge densities distributions originally centered in $\boldsymbol{x}$ and $\boldsymbol{y}$ respectively.

It can be shown [22] that the expansion in the 3D-h.o. basis of an interaction operator for an arbitrary central potential is of the form

$$\boldsymbol{H}^{q-q}_{Coulomb} = -\tfrac{1}{2}\sum_{\{N_i l_i j_i Y_i T_i\} J_0} V^{J_0}_{\{N_i l_i j_i Y_i T_i\}} \times \left[\left[\boldsymbol{b}^\dagger_{(N_1 l_1) j_1 Y_1 T_1} \otimes \boldsymbol{b}_{(N_2 l_2) j_2 \bar{Y}_2 T_2}\right]^{0 J_0 (11) 00} \otimes \left[\boldsymbol{b}^\dagger_{(N_3 l_3) j_3 Y_3 T_3} \otimes \boldsymbol{b}_{(N_4 l_4) j_4 \bar{Y}_4 T_4}\right]^{0 J_0 (11) 00}\right]^{00(00)0}_{0000} \ , \tag{9}$$

where the intermediate coupling refers to pseudospin 0, total angular momentum J_0, color irrep (11) and 00 in flavor isospin and hypercharge. On the other hand, the coefficients $V^{J_0}_{\{N_i l_i j_i Y_i T_i\}}$ are given by

$$\begin{aligned} V^{J_0}_{\{N_i l_i j_i Y_i T_i\}} = {} & (-1)^{\frac{1}{3}+\frac{Y_1}{2}+T_1}\tfrac{\sqrt{2T_1+1}}{\sqrt{3}}(-1)^{\frac{1}{3}+\frac{Y_3}{2}+T_3}\tfrac{\sqrt{2T_3+1}}{\sqrt{3}}\delta_{Y_1 Y_2}\delta_{T_1 T_2}\delta_{Y_3 Y_4}\delta_{T_3 T_4} \\ & \hat{j}_1\hat{j}_2\hat{j}_3\hat{j}_4\hat{J}_0 \sum_S \sum_{N'_r N_r l_r N_R l_R J} (-1)^{j_2+j_3+J_0+J}(\hat{l}_r)^2(\hat{J})^2(\hat{S})^2 \\ & \left\{\begin{array}{ccc} j_1 & j_3 & J \\ j_2 & j_4 & J_0 \end{array}\right\} \left\{\begin{array}{ccc} l_1 & \frac{1}{2} & j_1 \\ l_3 & \frac{1}{2} & j_3 \\ l_r & S & J \end{array}\right\} \left\{\begin{array}{ccc} l_2 & \frac{1}{2} & j_2 \\ l_4 & \frac{1}{2} & j_4 \\ l_r & S & J \end{array}\right\} \\ & \langle N'_r l_r, N_R l_R; l_r | N_1 l_1, N_3 l_3; l_r\rangle \langle N_r l_r, N_R l_R; l_r | N_2 l_2, N_4 l_4; l_r\rangle \\ & \int_0^\infty r^2 dr R_{n'_r l_r}(r) V(r) R_{n_r l_r}(r) \ , \end{aligned} \tag{10}$$

where $\hat{j}$ is the shorthand notation for $\hat{j} = \sqrt{2j+1}$. In particular, for the Cornell Potential $V(r) = -\frac{\alpha}{r} + \beta r$, the coefficients $V^{J_0}_{\{N_i l_i j_i Y_i T_i\}}$ involve only integrals of the form .

$$I_{\{n_i, l_i, L\}} = \int R^2 dR \int r^2 dr R_{n_1 l_1}(r) R_{n_2 l_2}(r) \left(\frac{\alpha}{r} + \beta r\right) R_{n_3 l_3}(R) R_{n_4 l_4}(R) \tag{11}$$

which are called *Talmi Integrals* and are analytical and well behaved in both the limits when $r \to 0$ and $r \to \infty$. In addition, the dependence on the CM coordinate R sums out naturally

through the orthogonality rules of the TM coefficients, so that the interaction does not generate spurious excitations.

After applying the transformation in (6), each of the intermediate couplings of the equation (9) as can be rewritten as

$$\begin{aligned}
&\left[\boldsymbol{Q}^{\dagger}_{\tau_1(N_1l_1)j_1Y_1T_1}\otimes\boldsymbol{Q}_{\tau_2(N_2l_2)j_2\bar{Y}_2T_2}\right]^{0J_0(11)00}=\tfrac{1}{\sqrt{2}}\textstyle\sum_{\tau\pi_1k_1\pi_2k_2}\\
&\times\Bigg\{\left(\alpha^{j_1}_{\tau(N_1l_1)\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{b}^{\dagger}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{b}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\\
&-\left(\alpha^{j_1}_{\tau(N_1l_1)\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)-\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{b}^{\dagger}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{d}^{\dagger}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\\
&+\left(\alpha^{j_1}_{\tau(N_1l_1)-\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{d}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{b}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\\
&-\left(\alpha^{j_1}_{\tau(N_1l_1)-\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)-\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{d}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{d}^{\dagger}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\Bigg\}\quad,
\end{aligned}\tag{12}$$

where the sum over the pseudo-spin λ_1,λ_2 has been written explicitly in order to trace the type of excitation it represents.

To express in a simple manner the result of applying the transformation (6) to the operators involved in the interaction, it turns out convenient to introduce the short notation for operators $\mathcal{A}$ and $\mathcal{B}$, which include only the couplings in the $SU(2)$ subalgebra of flavor.

$$\begin{aligned}
\mathcal{A}^{\tilde{\Gamma}}_{\pi_1\mathbf{1}_1\pi_2\bar{\mathbf{1}}_2;\,\tilde{\mu}} &= \frac{1}{\sqrt{2}}\left(\alpha^{j_1}_{\tau(N_1l_1)\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{b}^{\dagger}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{b}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\\
&- \frac{1}{\sqrt{2}}\left(\alpha^{j_1}_{\tau(N_1l_1)-\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)-\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{d}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{d}^{\dagger}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\\
\mathcal{B}^{\tilde{\Gamma}}_{\pi_1\mathbf{1}_1\pi_2\bar{\mathbf{1}}_2;\,\tilde{\mu}} &= \frac{1}{\sqrt{2}}\left(\alpha^{j_1}_{\tau(N_1l_1)-\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{d}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{b}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\\
&- \frac{1}{\sqrt{2}}\left(\alpha^{j_1}_{\tau(N_1l_1)\frac{1}{2}\pi_1k_1}\right)^{*}\left(\alpha^{j_2}_{\tau(N_2l_2)-\frac{1}{2}\pi_2k_2}\right)\left[\boldsymbol{b}^{\dagger}_{\pi_1k_1j_1Y_1T_1}\otimes\boldsymbol{d}^{\dagger}_{\pi_2k_2j_2\bar{Y}_2T_2}\right]^{J_0(11)00}_{MC}\quad.
\end{aligned}\tag{13}$$

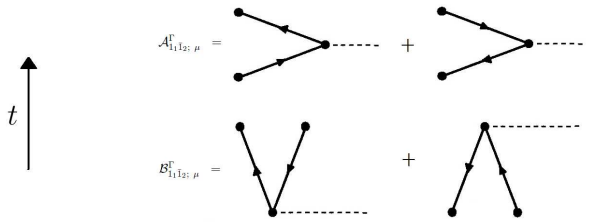

Figure 1. The $\mathcal{A}$ operator involves dispersion diagrams while the $\mathcal{B}$ operator represents the creation or annihilation of a pair.

In this way, the interaction can be expressed in a simple form as

$$\begin{aligned}
&\boldsymbol{H}_{coul}=-\tfrac{1}{2}\textstyle\sum_{J_0\tau\tau'}\sum_{\{N_il_ik_ij_i\lambda_i\pi_iY_iT_i\}}V'^{J_0}_{\{N_il_ij_iY_iT_i\}}\\
&\times\Big(\,[\mathcal{A}_{12}\mathcal{A}_{34}]^0_0+[\mathcal{A}_{12}\mathcal{B}_{34}]^0_0+[\mathcal{B}_{12}\mathcal{A}_{34}]^0_0+[\mathcal{B}_{12}\mathcal{B}_{34}]^0_0\Big)\,.
\end{aligned}\tag{14}$$

One can notice that only the combinations $[\mathcal{A}_{12}\mathcal{A}_{34}]$ and $[\mathcal{B}_{12}\mathcal{B}_{34}]$ will have a contribution at both the TDA and RPA level and thus, in this approach one can consider the Hamiltonian of the model to be

$$\begin{aligned} \boldsymbol{H} = \textstyle\sum_{kj} \epsilon_{kjYT} \left(\boldsymbol{b}^{\dagger}_{kj} \cdot \boldsymbol{b}^{kj} - \boldsymbol{d}_{kj} \cdot \boldsymbol{d}^{\dagger kj} \right) \\ -\tfrac{1}{2} \textstyle\sum_{J_0 \tau \tau'} \sum_{\{N_i l_i k_i j_i \lambda_i \pi_i Y_i T_i\}} V^{J_0}_{\{N_i l_i j_i Y_i T_i\}} \Big([\mathcal{A}_{12}\mathcal{A}_{34}]^0_0 + [\mathcal{B}_{12}\mathcal{B}_{34}]^0_0 \Big) \ , \end{aligned} \tag{15}$$

where the coefficients $V^{J_0}_{\{N_i l_i j_i Y_i T_i\}}$ have an analytical expression for the Cornell Potential [22].

4. The diagonalization method

The effective Hamiltonian of the model presented in equation (15) consists of a one-body free propagation term and a two-body interaction mediated by the Cornell static potential. This kind of non-relativistic Hamiltonians of at most two-body operators is a standard problem and there is a lot of literature with several methods of solution. In particular, the Many-body Methods have shown to be a powerful tool in describing quantum systems where interactions between many particles create strong quantum correlations between them.

As mentioned in Section 2, in the present model we have chosen to use two different many-body methods in order to diagonalize our Hamiltonian: the TDA and the RPA methods. The first one does not include ground state correlations, so it would correspond to the vacuum of a free theory and therefore is a naive approximation to a strong interacting theory such as NP-QCD. Pair correlations in the vacuum are introduced by the RPA method and thus only such calculation are reported in Section 5. For further details on the comparison of both methods, see [22].

The main reason for applying the many-body methods relies in the fact that the physical interpretation is direct: On one hand, in the language of nuclear physics any low-energy excitation of the system can be interpreted as the creation of a particle above the Fermi level of the nucleai and a hole below it. On the other hand, in Dirac's scheme, fermionic antimatter can be interpreted as states with negative energy that completely fill the Dirac Sea up to a Fermi energy $\epsilon_F = 0$, so that in a direct analogy to nuclear physics, one can interpret a quark-antiquark excitation as the creation of a pseudo-particle above the Dirac Sea and the creation of its respective hole below it. This allows us to model the light-flavored mesons as pairs that are scalar in color.

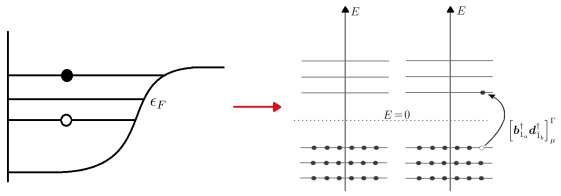

Figure 2. On the left, a particle-hole excitation above the nucleai's Fermi level. On the right, a quark-antiquark excitation above the Dirac sea.

Another reason for using this kind of methods is based in the idea that the appearance of a quark-antiquark vacuum condensate is typically associated with the confining potential that arises when fixing the Coulomb gauge [23, 24]. Thus, it is expected that the pairs involved in both the TDA and RPA methods form a suitable basis for describing the physics involved.

4.1. A posteriori flavor mixing

After applying the many-body methods, it is possible to identify η and η'-like mesons, as pure $q\bar{q}$ and $s\bar{s}$ states, which can be mixed by introducing an *a posteriori* interaction via

$$H_{mix} = \begin{pmatrix} M_{q\bar{q}} + 2H_{FM} & \sqrt{2}\, H_{FM} \\ \sqrt{2}\, H_{FM} & M_{s\bar{s}} + H_{FM} \end{pmatrix} . \tag{16}$$

The masses $M_{q\bar{q}}$ and $M_{s\bar{s}}$ of the equation (16) correspond to the lowest eigenvalues at the RPA level for the pure $q\bar{q}$ and $s\bar{s}$ states of the Hamiltonian (3) in the subspace $(J^P)YT = \{0^-, (0,0)\}$.

The interaction in (16) emulates the dynamical mechanism responsible for flavor mixing, which is the annihilation of a quark-antiquark pair into two or three (depending on the angular momentum conservation rules) gluons coupled to color zero that subsequently create a new quark-antiquark colorless pair, but not necessarily with the same flavor [17].

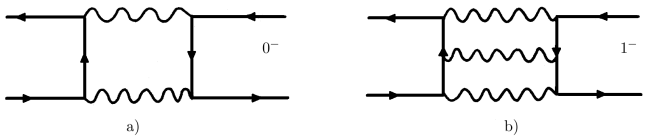

Figure 3. Virtual annihilation of neutral mesons into gluons for a) $J^P = 0^-$ and b) $J^P = 1^-$. Such process is responsible of the dynamical flavor mixing.

The eigenvalues of H_{mix} are given by

$$E_{\pm} = \frac{1}{2}\Big(M_{q\bar{q}} + M_{s\bar{s}} + 3H_{FM} \pm \sqrt{9H_{FM}^2 + 2H_{FM}(M_{q\bar{q}} - M_{s\bar{s}}) + (M_{q\bar{q}} - M_{s\bar{s}})^2}\ \Big) . \tag{17}$$

Even though H_{FM} should in principle be a function of the energy scale in accordance with asymptotic freedom, in the present case is set to a constant value that could be interpreted as a medium field approximation. This constant is selected in order that the mass of the η' meson is reproduced.

5. Results

Since at the moment no distinction is made between charged and neutral mesons, only a pion-like state is reported, as well as a single kaon-like state.

Here, the parameters for the masses, α and β are adjusted so that the RPA calculation reproduces as well as possible the mass of the π^0 meson ($\approx$ 135 MeV). The parameter H_{FM} is chosen to exactly reproduce the mass of the η', though the election could have been to fit the energy gap between the η and η' mesons.

Without a flavor mixing interaction the pion-like states are degenerated with the η-like state. This would mean that the η state would be too far below its experimental value. With a flavor mixing interaction the η meson is still too low, but in better agreement with the experiments.

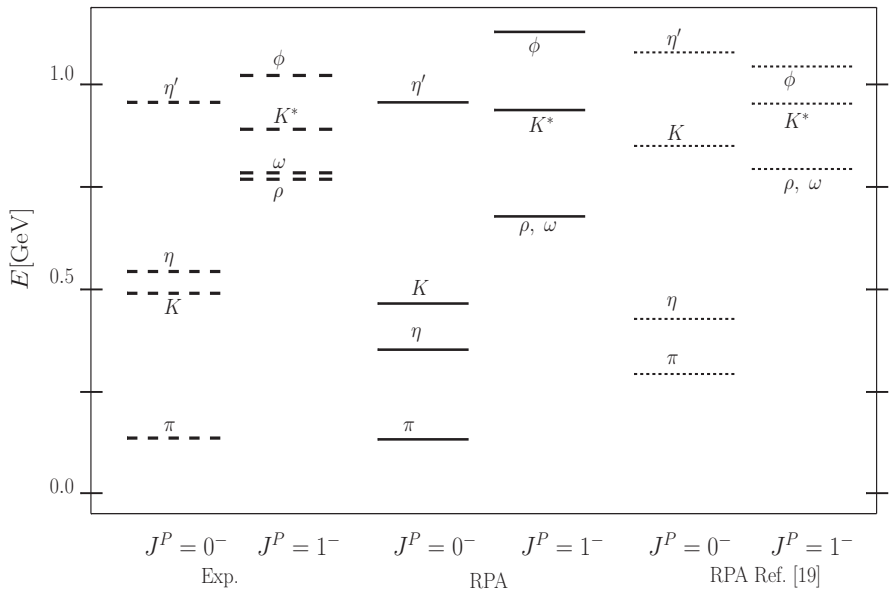

Figure 4. RPA pseudoscalar and vector states after adding the flavor mixing interaction. On the left, the experimental reported values for pseudoscalar and vector mesons up to 1 GeV. In the middle, the results for the present model. To the right, the RPA calculations for a plane wave basis from reference [19].

$m_{u,d}[GeV]$	$m_s[GeV]$	α	$\beta[GeV^2]$	H_{FM}[GeV]
0.05	0.31	0.16	0.40	0.188

Table 1. Parameters used for the RPA calculations shown on Figure 4 after applying the flavor mixing procedure in eq. (16). The optimal oscillator lenght corresponds to 0.75 fm

Last but not least, the spectrum for the pseudoscalar mesons obtained within the present model is in better agreement with the experiment than the results reported in [19], with particularly good results for the K and K^*-like states; while the rest of the spectrum is also relatively well reproduced.

6. Conclusions

The model is not ready yet and a lot of improvements to the spectra shown in Figure 4 can be achieved, for example, by introducing a dynamical flavor mixing.

The recovery of the traslational invariance of the CM with the use of the TM transformations is left for a future work and is expected to greatly improve the current results.

The model's advantages obtained so far are summarized below:

- It is a simple yet semi-realistic model. Even though the spectra can be improved, it's current state is relatively good.
- The 3D-h.o. basis is confining and allows analytic integration, as the integrals involved in the interaction matrix elements are well behaved.
- Rapid convergence of the solutions respect to the number of quanta considered. By introducing a remormalization scheme [22], the solutions barely change after $N_{max} = 8$.

- The calculations involved do not require too much computational power. This is both because of the rapid convergence of the solutions and due to the fact that the integrals are precalculated analically, as mentioned above.
- The coordinate space does not contain any divergences as those found on momentum space for plane wave calculations.
- Unlike lattice QCD, the space-time is intrinsically continuous, introducing no inconvenients due to lattice finite spacing.
- The structure of a correlated vacuum introduced by the RPA method simulates the virtual pairs creation.

Because of the above, it can be concluded that the present model is feasible but still requires further research to be able to compete with the current approaches for NP-QCD calculations.

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