

## ON ROOTED DIRECTED PATH GRAPHS

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**ABSTRACT.** An asteroidal triple is a stable set of three vertices such that each pair is connected by a path avoiding the neighborhood of the third vertex. An asteroidal quadruple is a stable set of four vertices such that any three of them is an asteroidal triple.

Two non adjacent vertices are linked by a special connection if either they have a common neighbor or they are the endpoints of two vertex-disjoint chordless paths satisfying certain technical conditions. Cameron, Hoàng, and Lévêque [DIMAP Workshop on Algorithmic Graph Theory, 67–74, Electron. Notes Discrete Math., 32, Elsevier, 2009] proved that if a pair of non adjacent vertices are linked by a special connection then in any directed path model  $T$  the subpaths of  $T$  corresponding to the vertices forming the special connection have to overlap and they force  $T$  to be completely directed in one direction between these vertices. Special connections along with the concept of asteroidal quadruple play an important role to study rooted directed path graphs, which are the intersection graphs of directed paths in a rooted directed tree.

In this work we define other special connections; these special connections along with the ones defined by Cameron, Hoàng, and Lévêque are nine in total, and we prove that every one forces  $T$  to be completely directed in one direction between these vertices. Also, we give a characterization of rooted directed path graphs whose rooted models cannot be rooted on a bold maximal clique. As a by-product of our result, we build new forbidden induced subgraphs for rooted directed path graphs.

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### 1. INTRODUCTION

A graph is *chordal* if it contains no cycle of length at least four as an induced subgraph. A classical result [4] states that a graph is chordal if and only if it is the (vertex) intersection graph of a family of subtrees of a tree.

Natural subclasses of chordal graphs are path graphs, directed path graphs, rooted directed path graphs and interval graphs. A graph is a *path graph* if it is the intersection graph of a family of subpaths of a tree. A graph is a *directed path graph* if it is the intersection graph of a family of directed subpaths of a directed tree. A graph is a *rooted path graph* if it is the intersection graph of a family of directed subpaths of a rooted tree. A graph is an *interval graph* if it is the intersection graph of a family of subpaths of a path.

By definition we have the following inclusions between the different classes considered (and these inclusions are strict):  $\text{interval} \subset \text{rooted directed path} \subset \text{directed path} \subset \text{path} \subset \text{chordal}$ .

Lekkerkerker and Boland [5] proved that a chordal graph is an interval graph if and only if it contains no asteroidal triple. As a by-product, they found a characterization of interval graphs by forbidden induced subgraphs.

Panda [8] found a characterization of directed path graphs by forbidden induced subgraphs, and then Cameron, Hoáng and Lévêque [2] gave a characterization of directed paths graph in terms of forbidden asteroids. For this purpose, they introduce the concept of a special connection. Two non adjacent vertices are linked by a special connection if they have a common neighbor or they are the endpoints of two vertex-disjoint paths of length three satisfying certain technical conditions. Special connections are interesting when considering directed path graphs because if  $a$  and  $b$ , two non adjacent vertices of a directed path graph, are linked by a special connection, then in every directed path model, the subpaths of  $T$  corresponding to the vertices forming the special connection have to overlap and they force  $T$  to be completely directed in one direction between  $a$  and  $b$ .

Clearly, rooted directed path graphs contain no asteroidal quadruples linked by special connections. The converse was conjectured by Cameron, Hoáng and Lévêque, but in this original form the conjecture is incomplete, since they could not describe all the connections between two non adjacent vertices that force any tree to be completely directed in one direction between these vertices.

In this article we define some special connections, which along with the ones defined in [1] are nine in total, and we prove that each one forces  $T$  to be completely directed in one direction between these vertices. Therefore, if  $a_1, a_2, a_3$  is a strong asteroidal and there is a special connection between  $a_1$  and  $a_2$  then no directed path model can be rooted on a maximal clique that contains  $a_3$ . Furthermore, we prove that the converse is true in case of leafage three, i.e., if the model cannot be rooted on a maximal clique that contains  $a_3$  then one of the nine special connection links  $a_1$  and  $a_2$ . As a by-product of our result, we build new forbidden induced subgraphs for rooted directed path graphs.

The paper is organized as follows: in Section 2 we give some definitions and background. In Section 3 we define special connections and prove that if a pair of non adjacent vertices are linked by a special connection, then in any directed path model  $T$  the subpaths of  $T$  corresponding to the vertices forming the special connection have to overlap and they force  $T$  to be completely directed in one direction between these vertices. In section 4 we give some properties of models that cannot be rooted on a bold maximal clique. Finally, in Section 5 we prove that  $G$  is a directed path graph with leafage three, and it has a strong asteroidal triple  $a_1, a_2, a_3$  such that there is a special connection between  $a_1$  and  $a_2$  if and only if no directed path model can be rooted on the maximal clique that contains  $a_3$ .

## 2. DEFINITIONS AND BACKGROUND

If  $G$  is a graph and  $V' \subseteq V(G)$ , then  $G \setminus V'$  denotes the subgraph of  $G$  induced by  $V(G) \setminus V'$ . If  $E' \subseteq E(G)$ , then  $G - E'$  denotes the subgraph of  $G$  induced by  $E(G) \setminus E'$ . If  $G, G'$  are two graphs, then  $G + G'$  denotes the graph whose vertices

are  $V(G) \cup V(G')$  and edges are  $E(G) \cup E(G')$ . Note that if  $T, T'$  are two trees such that  $|V(T) \cap V(T')| = 0$ , then  $T + T'$  is a forest.

A clique in a graph  $G$  is a set of pairwise adjacent vertices. Let  $\mathcal{C}(G)$  be the set of all maximal cliques of  $G$ .

The *neighborhood* of a vertex  $x$  is the set  $N(x)$  of vertices adjacent to  $x$  and the *closed neighborhood* of  $x$  is the set  $N[x] = \{x\} \cup N(x)$ . A vertex is *simplicial* if its closed neighborhood is a maximal clique. Two adjacent vertices  $x$  and  $y$  are *twins* if  $N[x] = N[y]$ .

A *strong asteroidal* of a graph  $G$  is a stable set  $\{a_1, \dots, a_n\}$  ( $n \geq 2$ ) of vertices of  $G$  such that  $G \setminus N[a_i]$  is a connected graph for  $i = 1, \dots, n$ .

A *clique tree*  $T$  of a graph  $G$  is a tree whose vertices are the elements of  $\mathcal{C}(G)$  and such that for each vertex  $x$  of  $G$ , those elements of  $\mathcal{C}(G)$  that contain  $x$  induce a subtree of  $T$ , which we will denote by  $T_x$ . Note that  $G$  is the intersection graph of the subtrees  $(T_x)_{x \in V(G)}$ . In this paper, whenever we talk about the intersection of subgraphs of a graph we mean that the vertex sets of the subgraphs intersect.

Given two non adjacent vertices  $a, b$  of  $G$ , and a clique tree  $T$  of  $G$ ,  $T(a, b)$  is defined as the subtree of  $T$  of minimum size that contains at least a vertex of  $T_a$  and  $T_b$ .

Gavril [4] proved that a graph is chordal if and only if it has a clique tree. Clique trees are called *models* of the graph.

Observe that if  $G$  has a strong asteroidal  $a_1, \dots, a_n$  then every clique tree has  $N[a_i]$  as a leaf for  $i = 1, \dots, n$ . So  $a_i$  is a simplicial vertex of  $G$  for  $i = 1, \dots, n$ .

In [7] Monma and Wei introduced the notation UV, DV and RDV to refer to the classes of path graphs, directed path graphs and rooted directed path graphs, respectively. They also proved the following clique tree characterizations for these classes. A graph is a *path graph* or a *UV graph* if it admits a *UV-model*, i.e., a clique tree  $T$  such that  $T_x$  is a subpath of  $T$  for every  $x \in V(G)$ . A graph is a *directed path graph* or a *DV graph* if it admits a *DV-model*, i.e., a clique tree  $T$  whose edges can be directed such that  $T_x$  is a directed subpath of  $T$  for every  $x \in V(G)$ . A graph is a *rooted path graph* or an *RDV graph*, if it admits an *RDV-model*, i.e., a clique tree  $T$  that can be rooted and whose edges are directed from the root toward the leaves such that  $T_x$  is a directed subpath of  $T$  for every  $x \in V(G)$ .

It has been proved in [3] that if  $G$  is a DV-graph, then any UV-model of  $G$  can be directed to obtain a DV-model of  $G$ . We say that a DV-model  $T$  of a DV graph  $G$  can be *rooted* if  $T$  can be rooted on a vertex such that it becomes an RDV-model of  $G$ .

Let  $T$  be a clique tree. We often use capital letters to denote the vertices of a clique tree as these vertices correspond to maximal cliques of  $G$ . In order to simplify the notation, we often write  $X \in T$  instead of  $X \in V(T)$ , and  $e \in T$  instead of  $e \in E(T)$ . If  $T'$  is a subtree of  $T$ , then  $G_{T'}$  denotes the subgraph of  $G$  that is induced by the vertices of  $\cup_{X \in V(T')} X$ .

Let  $T$  be a tree. For  $V' \subseteq V(T)$ , let  $T[V']$  be the minimal subtree of  $T$  containing  $V'$ . Then for  $X, Y \in V(T)$ ,  $T[X, Y]$  is the subpath of  $T$  between  $X$  and  $Y$ . Let  $T[X, Y) = T[X, Y] \setminus Y$ ,  $T(X, Y) = T[X, Y] \setminus X$ , and  $T(X, Y) = T[X, Y] \setminus \{X, Y\}$ .

Note that some of these paths may be empty or reduced to a single vertex when  $X$  and  $Y$  are equal or adjacent. If  $X \in V(T)$  and  $e \in E(T)$  with  $e = AB$  and  $A \in T[X, B]$ , then let  $T[X, e] = T[X, B]$ ,  $T[X, e] = T[X, A]$ ,  $T(X, e) = T(X, B)$  and  $T(X, e) = T(X, A)$ . Given a vertex  $X \in V(T(Y, Z))$ , we say that there is a *vertex crossing*  $X$  in  $T[Y, Z]$  if  $X' \cap X'' \neq \emptyset$ , where  $X'$  and  $X''$  are the two neighbors of  $X$  in  $T[Y, Z]$ .

Let  $T$  be a tree, we denote by  $\ln(T)$  the number of leaves of  $T$ . The *leafage* of a chordal graph  $G$ , denoted by  $\ell(G)$ , is the minimum integer  $k$  such that  $G$  admits a model  $T$  with  $\ln(T) = k$ . Note that if  $G$  has a strong asteroidal  $a_1, \dots, a_n$  then  $\ell(G) \geq n$ .

In a clique tree  $T$ , the *label* of an edge  $AB$  of  $T$  is defined as  $lab(AB) = A \cap B$ . We will say that  $e, e'$  in the same clique tree  $T$  are *twin edges* if  $lab(e) = lab(e')$ .

Let  $T$  be a DV-model of  $G$ , let  $Q$  be a vertex of  $T$ , and let  $e$  be an edge of  $T$ . Let  $T_1$  and  $T_2$  be the two connected components of  $T - e$  where  $Q$  is in  $T_1$ . We say that vertices in  $lab(e)$  have *the same end* with respect to  $Q$  if there exists a vertex  $Q'$  in  $T_1$ , possibly  $Q' = Q$ , such that for each  $x \in lab(e)$ , one endpoint of  $T_x$  is  $Q'$ .

We say that  $X \in V(T)$  *dominates*  $e \in E(T)$  if  $lab(e) \subseteq X$ . On the other hand, an edge  $e$  satisfying a given property  $P$  is *maximally farthest from a vertex*  $C$  if there is no edge  $e'$ , different from  $e$ , satisfying this property and such that  $e$  is between  $C$  and  $e'$ .

### 3. SPECIAL CONNECTIONS

Let  $a$  and  $b$  be two non adjacent vertices of a graph  $G$ . We will define nine *types of connection* between these vertices. Observe that Types 1, 2 and 3 were already defined in [2].

- **Type 1:** there exists a path  $P = a, x, b$  in  $G$ .
- **Type 2:** there exist two paths  $P = a, y_1, y_2, b$  and  $Q = a, x_1, x_2, b$  in  $G$  such that  $\{x_1, y_1, y_2\}$  and  $\{x_1, x_2, y_2\}$  are cliques of  $G$ .
- **Type 3:** there exist two paths  $P = a, y_1, y_2, b$ ,  $Q = a, x_1, x_2, b$ , and two vertices  $s_1, s_2$  in  $G$  such that  $\{x_1, x_2, y_1, y_2\}$ ,  $\{x_1, y_1, y_2, s_1\}$  and  $\{x_1, x_2, y_2, s_2\}$  are cliques of  $G$ . In this case it is said that  $\{x_1, x_2, y_1, y_2, s_1, s_2\}$  induces an antenna.

Next we define new special connections.

- **Type 4:** there exist two paths  $P = a, y_1, y_2, b$ ,  $Q = a, x_1, x_2, b$ , and vertices in  $G$ :  $t, u, z_i$  for  $i \in \{1, \dots, o\}$  such that  $\{x_1, x_2, y_1, y_2, z_o\}$ ,  $\{x_1, y_1, z_i, z_{i+1}\}_{i=1, \dots, o-1}$ ,  $\{y_1, z_1, u\}$  and  $\{x_1, x_2, y_1, t\}$  are cliques of  $G$  (Figure 1a).
- **Type 5:** there exist two paths  $P = a, y_1, y_2, b$ ,  $Q = a, x_1, x_2, b$ , and vertices in  $G$ :  $s_1, s, t, t_i$  for  $i \in \{1, \dots, p\}$ ,  $u, z_i$  for  $i \in \{1, \dots, o\}$  such that  $\{x_1, x_2, y_1, y_2, z_o, s_1\}$ ,  $\{x_1, x_2, y_1, y_2, z_o, t_p\}$ ,  $\{x_1, x_2, y_1, z_o, t_p, s\}$ ,  $\{x_1, y_1, z_i, z_{i+1}\}_{i=1, \dots, o-1}$ ,  $\{y_1, z_1, u\}$ ,  $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1, \dots, p-1}$  and  $\{x_1, y_1, t_1, t\}$  are cliques of  $G$  (Figure 1b).
- **Type 6:** there exist two paths  $P = a, y_1, y_2, b$ ,  $Q = a, x_1, x_2, b$ , and vertices in  $G$ :  $t, t_i$  for  $i \in \{1, \dots, p\}$ ,  $u, z_i$  for  $i \in \{1, \dots, o\}$  such that  $\{x_1, x_2, y_1, y_2, z_o\}$ ,  $\{x_1, x_2, y_1, z_o, t_p\}$ ,  $\{x_1, y_1, z_i, z_{i+1}\}_{i=1, \dots, o-1}$ ,  $\{x_1, y_1, t_1, t\}$

$z_o, t_i, t_{i+1}\}_{i=1, \dots, p-1}$ ,  $\{y_1, z_1, u\}$  and  $\{x_1, y_1, t_1, t\}$  are cliques of  $G$  (Figure 1c).

- **Type 7:** there exist two paths  $P = a, y_1, y_2, b$ ,  $Q = a, x_1, x_2, b$ , and vertices in  $G$ :  $s, t, t_i$  for  $i \in \{1, \dots, p\}$ ,  $u, u', z_i$  for  $i \in \{1, \dots, o\}$ ,  $z'_i$  for  $i \in \{1, \dots, q\}$  such that  $\{x_1, x_2, y_1, y_2, z_o, t_p, z'_q\}$ ,  $\{x_1, x_2, y_1, z_o, t_p, s\}$ ,  $\{y_1, z_1, u\}$ ,  $\{x_1, y_1, z_o, z'_i, z'_{i+1}, t_p\}_{i=1, \dots, q-1}$ ,  $\{x_1, y_1, z_o, z'_1, u'\}$ ,  $\{x_1, y_1, z_i, z_{i+1}\}_{i=1, \dots, o-1}$ ,  $\{x_1, y_1, t_1, t\}$  and  $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1, \dots, p-1}$  are cliques of  $G$  (Figure 1d).
- **Type 8:** there exist two paths  $P = a, y_1, y_2, b$ ,  $Q = a, x_1, x_2, b$ , and vertices in  $G$ :  $t, t', t_i$  for  $i \in \{1, \dots, p\}$ ,  $t'_i$  for  $i \in \{1, \dots, r\}$ ,  $u, u', z_i$  for  $i \in \{1, \dots, o\}$ ,  $z'_i$  for  $i \in \{1, \dots, q\}$  such that  $\{x_1, x_2, y_1, y_2, z_o, t_p, z'_q\}$ ,  $\{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r\}$ ,  $\{x_1, y_1, z_o, t_p, t'_1, t'\}$ ,  $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1, \dots, p-1}$ ,  $\{y_1, z_1, u\}$ ,  $\{x_1, y_1, z_o, z'_q, t_p, t'_i, t'_{i+1}\}_{i=1, \dots, r-1}$ ,  $\{x_1, y_1, z_o, z'_1, u'\}$ ,  $\{x_1, y_1, z_i, z_{i+1}\}_{i=1, \dots, o-1}$ ,  $\{x_1, y_1, t_1, t\}$ , and  $\{x_1, y_1, z_o, t_p, z'_i, z'_{i+1}\}_{i=1, \dots, q-1}$  are cliques of  $G$  (Figure 1e).
- **Type 9:** there exist two paths  $P = a, y_1, y_2, b$ ,  $Q = a, x_1, x_2, b$ , and vertices in  $G$ :  $s, s_1, t, t', t_i$  for  $i \in \{1, \dots, p\}$ ,  $t'_i$  for  $i \in \{1, \dots, r\}$ ,  $u, u', z_i$  for  $i \in \{1, \dots, o\}$ ,  $z'_i$  for  $i \in \{1, \dots, q\}$  such that  $\{x_1, x_2, y_1, y_2, z_o, t_p, z'_q, t'_r, s_1\}$ ,  $\{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r, s\}$ ,  $\{x_1, y_1, z_o, t_p, t'_1, t'\}$ ,  $\{x_1, y_1, z_o, t_i, t_{i+1}\}_{i=1, \dots, p-1}$ ,  $\{x_1, y_1, z_o, z'_q, t_p, t'_i, t'_{i+1}\}_{i=1, \dots, r-1}$ ,  $\{x_1, y_1, t_1, t\}$ ,  $\{x_1, y_1, z_i, z_{i+1}\}_{i=1, \dots, o-1}$ ,  $\{y_1, z_1, u\}$ ,  $\{x_1, y_1, z_o, z'_1, u'\}$  and  $\{x_1, y_1, z_o, t_p, z'_i, z'_{i+1}\}_{i=1, \dots, q-1}$  are cliques of  $G$  (Figure 1f).

**Theorem 1.** *Let  $G$  be a DV graph, and let  $a, b$  be two non adjacent vertices of  $G$  that are linked by Type  $i$  with  $1 \leq i \leq 9$ . Then, for every  $T$ , DV-model of  $G$ , the subpath  $T(a, b)$  is a directed path.*

*Proof.* Let  $Q_a$  be a maximal clique that contains  $a$ , and  $Q_b$  be a maximal clique that contains  $b$ .

- (1) Types 1, 2, 3. In [2] it was proved that if  $a$  and  $b$  are linked by Type  $i$ , with  $i \in \{1, 2, 3\}$ , then  $T[Q_a, Q_b]$  is a directed path of  $T$ .
- (2) We can assume that there is a special connection of Type 4, 5, 6, 7, 8 or 9.

The edge of  $T[Q_a, Q_b]$  incident to  $Q_a$  must have in its label  $S$  at least one vertex of  $\{x_1, x_2\}$  ( $\{y_1, y_2\}$ ), otherwise  $a$  and  $b$  are in two different components of  $G \setminus S$  contradicting that  $a, x_1, x_2, b$  ( $a, y_1, y_2, b$ ) is a path. Analogously, the edge incident to  $Q_b$  must have in its label at least one vertex of  $\{y_1, y_2\}$  ( $\{x_1, x_2\}$ ). Vertex  $a$  is not adjacent to  $x_2$ , and  $b$  is not adjacent to  $y_1$  then  $Q_a \cap \{x_1, x_2, y_1, y_2\} = \{x_1, y_1\}$  and  $Q_b \cap \{x_1, x_2, y_1, y_2\} = \{x_2, y_2\}$ .

(a) Suppose that the connection is of Type 4.

Let  $Q', Q_o, Q$  and  $Q_i$  be maximal cliques of  $G$  such that  $Q' \supset \{x_1, x_2, y_1, t\}$ ,  $Q_o \supset \{x_1, x_2, y_1, y_2, z_o\}$ ,  $Q \supset \{y_1, z_1, u\}$ , and  $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$  for  $i = 1, \dots, o - 1$ .

We will prove that  $Q_a, Q', Q_o, Q_b$  appear in this order in  $T$ .

As  $a$  is not adjacent to  $x_2$ ,  $b$  is not adjacent to  $x_1$ , and  $x_1, x_2$  are vertices in  $Q_o \cap Q'$ , then we have  $Q_a \notin T[Q_o, Q']$  and  $Q_b \notin T[Q_o, Q']$ . Observe that  $x_1$  and  $y_1$  are vertices in  $(Q' \cap Q_a) - (Q_o \cap Q_b)$ ,  $x_2$  and

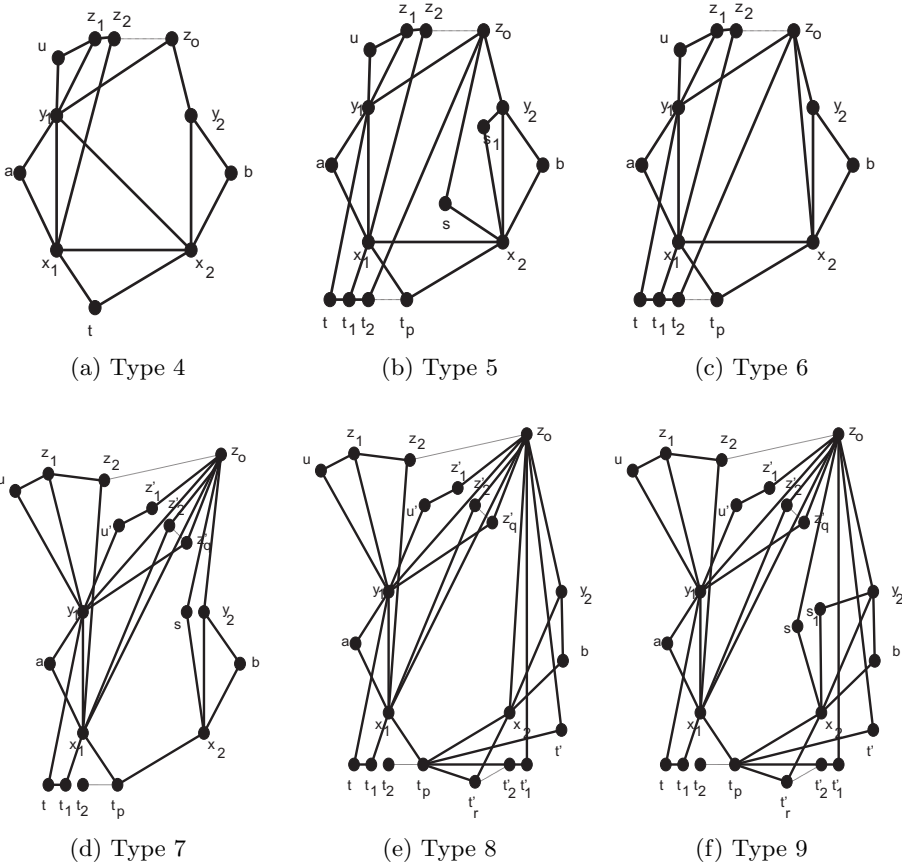


FIGURE 1. Types 4 through 9. We leave out edges of cliques of size greater than or equal to four.

$y_2$  are in  $(Q_o \cap Q_b) - (Q' \cap Q_a)$ ,  $x_2 \in Q' \cap Q_b$ , but neither  $x_1$  or  $y_1$  or  $y_2$  are in  $Q' \cap Q_b$ . Thus  $Q_a, Q', Q_o, Q_b$  appear in this order in  $T$ . Since  $x_1 \in Q_a \cap Q_o$  and  $x_2 \in Q' \cap Q_b$  it follows that  $T[Q_a, Q_b]$  is a directed path of  $T$ .

On the other hand  $Q_i \notin T[Q', Q_o]$  for  $i \neq o$ , since  $x_2$  is not adjacent to  $z_i$  for  $i \neq o$ . As  $x_1, y_1$  and  $z_i$  are vertices in  $Q_i \cap Q_{i+1}$  for  $i = 1, \dots, o-1$ ,  $x_1$  and  $y_1$  are in  $Q_i \cap Q_a$ , and  $z_1, y_1$  are vertices in  $Q_1 \cap Q$ , then  $Q_a, Q', Q_o, Q_{o-1}, \dots, Q_1, Q$  appear in this order in  $T$ . Vertex  $b$  is not adjacent to  $y_1$ , so  $Q_b \notin T[Q_a, Q]$ .

(b) Suppose that the connection is of Type 5.

Let  $Q_o, Q'_o, Q'_p, Q_i, Q, Q'_i$  and  $Q'$  be maximal cliques of  $G$  such that  $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, s_1\}$ ,  $Q'_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p\}$ ,  $Q'_p \supset \{x_1, x_2, y_1, z_o, t_p, s\}$ ,  $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$  for  $i = 1, \dots, o-1$ ,  $Q \supset$

$\{y_1, z_1, u\}$ ,  $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$  for  $i = 1, \dots, p - 1$ , and  $Q' \supset \{x_1, y_1, t_1, t\}$ . We will prove that  $Q_a, Q', Q'_1, \dots, Q'_p, Q'_o, Q_o, Q_b$  appear in this order in  $T$ .

Vertex  $s_1$  is not adjacent to  $t_p$ , and  $s$  is not adjacent to  $y_2$ , then  $Q_o \notin T[Q'_o, Q'_p]$  and  $Q'_p \notin T[Q'_o, Q_o]$  respectively. Observe that  $x_1, x_2, y_1, z_o$  and  $t_p$  are vertices in  $Q'_o \cap Q'_p$  but  $y_2 \notin Q'_o \cap Q'_p$ . Also  $x_1, x_2, y_1, y_2$  and  $z_o$  are in  $Q'_o \cap Q_o$  but  $t_p \notin Q'_o \cap Q_o$ . Thus  $Q'_p, Q'_o, Q_o$  appear in this order in  $T$ . On the other hand,  $Q'_i \notin T[Q'_p, Q_o]$  for  $i \neq p$  since  $t_i$  is not adjacent to  $x_2$ . As  $\{x_1, y_1, z_o, t_i\} \subset Q'_i \cap Q'_{i+1}$  and  $\{x_1, y_1, z_o\} \subset Q'_i \cap Q_o$ , but  $t_i \notin Q_o$  for  $i \neq p$ , it follows that  $Q'_1, \dots, Q'_p, Q'_o, Q_o$  appear in this order in  $T$ . Note that  $Q' \notin T[Q'_1, Q_o]$  since  $z_o$  is not adjacent to  $t$ ;  $\{x_1, y_1, t_1\} \subset Q' \cap Q'_1$ ,  $x_1$  and  $y_1$  are in  $Q' \cap Q_o$  but  $t_1 \notin Q_o$ , so  $Q', Q'_1, \dots, Q'_p, Q'_o, Q_o$  appear in this order in  $T$ . Since  $a$  is not adjacent to  $t_i$  for  $i = 1, \dots, p$ , and is also not adjacent to  $x_2$  then  $Q_a \notin T[Q', Q_o]$ . Vertex  $b$  is not adjacent to  $t_i$  for  $i = 1, \dots, p$  and is also not adjacent to  $y_1$ , so  $Q_b \notin T[Q', Q_o]$ . As  $x_1, y_1 \in (Q_a \cap Q') - (Q_o \cap Q_b)$  and  $x_2, y_2 \in (Q_o \cap Q_b) - (Q_a \cap Q')$ , it follows that  $Q_a, Q', Q'_1, \dots, Q'_p, Q'_o, Q_o, Q_b$  appear in this order in  $T$ . Since  $x_1 \in Q_a \cap Q_o$  and  $x_2 \in Q' \cap Q_b$  then  $T[Q_a, Q_b]$  is a directed path of  $T$ .

Using the same argument of Case 2a,  $Q_a, Q', Q'_1, \dots, Q'_o, Q_o, Q_{o-1}, \dots, Q_1, Q$  appear in this order in  $T$  and also  $Q_b \notin T[Q_a, Q]$ .

(c) Suppose that the connection is of Type 6 or 7.

In case that the connection is of Type 6, let  $Q_o, Q'_p, Q_i, Q, Q'_i$  and  $Q'$  be maximal cliques of  $G$  such that  $Q_o \supset \{x_1, x_2, y_1, y_2, z_o\}$ ,  $Q'_p \supset \{x_1, x_2, y_1, z_o, t_p\}$ ,  $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$  for  $i = 1, \dots, o - 1$ ,  $Q \supset \{y_1, z_1, u\}$ ,  $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$  for  $i = 1, \dots, p - 1$ , and  $Q' \supset \{x_1, y_1, t_1, t\}$ .

In case that the connection is of Type 7, let  $Q_o, Q'_p, Q''_i, Q'', Q_i, Q'_i$  and  $Q'$  be maximal cliques of  $G$  such that  $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p, z'_q\}$ ,  $Q'_p \supset \{x_1, x_2, y_1, z_o, t_p, s\}$ ,  $Q''_i \supset \{x_1, y_1, z_o, z'_i, z'_{i+1}, t_p\}$  for  $i = 1, \dots, q - 1$ ,  $Q'' \supset \{x_1, y_1, z_o, z'_1, u'\}$ ,  $Q' \supset \{x_1, y_1, t_1, t\}$ ,  $Q \supset \{y_1, z_1, u\}$ ,  $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$  for  $i = 1, \dots, o - 1$ , and  $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$  for  $i = 1, \dots, p - 1$ .

In both cases, we will prove that  $Q_a, Q', Q'_1, \dots, Q'_p, Q_o, Q_b$  appear in this order in  $T$ .

We know that  $\{x_1, x_2, y_1, z_o, t_p\} \subset Q_o \cap Q'_p$  if the connection is of Type 7, and  $\{x_1, x_2, y_1, z_o\} \subset Q_o \cap Q'_p$  if the connection is of Type 6. But in both cases we have  $y_2 \notin Q_o \cap Q'_p$ . Vertex  $t_i$  is not adjacent to  $x_2$  for  $i \neq p$ , then  $Q'_i \notin T[Q_o, Q'_p]$ . Observe that  $\{x_1, y_1, z_o, t_i\} \subset Q'_i \cap Q'_{i+1}$ ,  $\{x_1, y_1, z_o\} \subset Q'_i \cap Q_o$  but  $t_i \notin Q_o$  for  $i \neq p$ . Thus  $Q'_1, \dots, Q'_p, Q_o$  appear in this order in  $T$ . Note that  $Q' \notin T[Q'_1, Q_o]$  since  $z_o$  is not adjacent to  $t$ . Vertices  $x_1, y_1$  and  $t_1$  are in  $Q' \cap Q'_1$ ,  $x_1$  and  $y_1$  are  $Q' \cap Q_o$  but  $t_1 \notin Q_o$ , so  $Q', Q'_1, \dots, Q'_p, Q_o$  appear in this

order in  $T$ . Vertex  $a$  is not adjacent to  $t_i$  for  $i = 1, \dots, p$ , and is also not adjacent to  $x_2$ , then  $Q_a \notin T[Q', Q_o]$ . Since  $b$  is not adjacent to  $t_i$  for  $i = 1, \dots, p$  and  $b$  is not adjacent to  $y_1$  so  $Q_b \notin T[Q', Q_o]$ . As  $x_1, y_1 \in (Q_a \cap Q') - (Q_o \cap Q_b)$  and  $x_2, y_2 \in (Q_o \cap Q_b) - (Q_a \cap Q')$ , then  $Q_a, Q', Q'_1, \dots, Q'_p, Q_o, Q_b$  appear in this order in  $T$ . Since  $x_1 \in Q_a \cap Q_o$  and  $x_2 \in Q' \cap Q_b$  it follows that  $T[Q_a, Q_b]$  is a directed path of  $T$ .

On the other hand, in case that the connection is of Type 6, using the same argument of Case 2a, we have that  $Q_a, Q', Q'_1, \dots, Q_o, Q_{o-1}, \dots, Q_1, Q$  appear in this order in  $T$ . And as  $b$  is not adjacent to  $y_1$  then  $Q_b \notin T[Q_a, Q]$ .

In case that the connection is of Type 7, as  $z'_i$  is not adjacent to  $z_j$  for  $j \in \{1, \dots, o-1\}$  and  $i \in \{1, \dots, q\}$  it follows that  $Q'''_i, Q''_i \notin T[Q_j, Q_{j+1}]$ . Also  $Q''_i \notin T[Q'_1, Q_o]$  since  $z'_i$  is not adjacent to  $t_j$  for  $i \in \{1, \dots, q-1\}$  and  $j \in \{1, \dots, p-1\}$ . Observe that  $\{x_1, y_1, z_o, t_p, z'_i\} \subset Q''_i \cap Q''_{i+1}$  for  $i \neq q$ ,  $\{x_1, y_1, z_o, t_p\} \subset Q''_i \cap Q_o$ ,  $\{x_1, y_1\} \subset Q_a \cap Q''_i$ , and  $\{x_1, y_1, z_o\} \subset Q''_i \cap Q_{o-1}$  then  $Q_a, Q_1, \dots, Q_o, Q'_q, \dots, Q'_1, Q_{o-1}, \dots, Q$  appear in this order in  $T$ , and also  $Q_b \notin T[Q_a, Q]$ .

(d) Suppose that the connection is of Type 8 or 9.

In case that the connection is of Type 8, let  $Q_o, Q'_p, Q^{iv}, Q'_i, Q'''_i, Q', Q'', Q, Q_i$ , and  $Q''_i$  be maximal cliques of  $G$  such that  $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p, z'_q\}$ ,  $Q'_p \supset \{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r\}$ ,  $Q^{iv} \supset \{x_1, y_1, z_o, t_p, t'_1, t'\}$ ,  $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$  for  $i = 1, \dots, p-1$ ,  $Q'''_i \supset \{x_1, y_1, z_o, z'_q, t_p, t'_i, t'_{i+1}\}$  for  $i = 1, \dots, r-1$ ,  $Q' \supset \{x_1, y_1, t_1, t\}$ ,  $Q'' \supset \{x_1, y_1, z_o, z'_1, u'\}$ ,  $Q \supset \{y_1, z_1, u\}$ ,  $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$  for  $i = 1, \dots, o-1$ , and  $Q''_i \supset \{x_1, y_1, z_o, t_p, z'_i, z'_{i+1}\}$  for  $i = 1, \dots, q-1$ .

In case that the connection is of Type 9, let  $Q_o, Q'_p, Q^{iv}, Q'_i, Q''_i, Q'''_i, Q', Q_i, Q$  and  $Q''_i$  be maximal cliques of  $G$  such that  $Q_o \supset \{x_1, x_2, y_1, y_2, z_o, t_p, z'_q, t'_r, s_1\}$ ,  $Q'_p \supset \{x_1, x_2, y_1, z_o, z'_q, t_p, t'_r, s\}$ ,  $Q^{iv} \supset \{x_1, y_1, z_o, t_p, t'_1, t'\}$ ,  $Q'_i \supset \{x_1, y_1, z_o, t_i, t_{i+1}\}$  for  $i = 1, \dots, p-1$ ,  $Q'' \supset \{x_1, y_1, z_o, z'_1, u'\}$ ,  $Q'''_i \supset \{x_1, y_1, z_o, z'_q, t_p, t'_i, t'_{i+1}\}$  for  $i = 1, \dots, r-1$ ,  $Q' \supset \{x_1, y_1, t_1, t\}$ ,  $Q_i \supset \{x_1, y_1, z_i, z_{i+1}\}$  for  $i = 1, \dots, o-1$ ,  $Q \supset \{y_1, z_1, u\}$ , and  $Q''_i \supset \{x_1, y_1, z_o, t_p, z'_i, z'_{i+1}\}$  for  $i = 1, \dots, q-1$ .

In both cases, we will prove that  $Q_a, Q', Q'_1, \dots, Q'_{p-1}, Q^{iv}, Q'''_1, \dots, Q'''_{r-1}, Q'_p, Q_o, Q_b$  appear in this order in  $T$ .

We know that  $\{x_1, x_2, y_1, z_o, t_p, z'_q\} \subset Q_o \cap Q'_p$  but  $y_2 \notin Q_o \cap Q'_p$ . As  $t'_i$  is not adjacent to  $x_2$  for  $i \neq r$ , so  $Q'''_i \notin T[Q_o, Q'_p]$ .

Observe that  $\{x_1, y_1, z_o, z'_q, t'_i\} \subset Q'''_i \cap Q'''_{i+1}$ ,  $\{x_1, y_1, z_o, z'_q, t_p, t'_r\} \subset Q'''_{r-1} \cap Q'_p$  but  $t'_r \notin Q_o$ ,  $\{x_1, y_1, z_o, z'_q, t_p\} \subset Q'''_i \cap Q'_p$  but  $t'_i \notin Q'_p$  for  $i \neq r$ . Then  $Q'''_1, \dots, Q'''_{r-1}, Q'_p, Q_o$  appear in this order in  $T$ .

On the other hand,  $Q^{iv} \notin T[Q'_1, Q_o]$  since  $z'_q$  is not adjacent to  $t'_1$ . Vertices  $x_1, y_1, z_o, t_p$  and  $t'_1$  are in  $Q^{iv} \cap Q'''_1$ ,  $x_1, y_1, z_o$  and  $t_p$  are in  $Q^{iv} \cap Q_o$  but  $t'_1 \notin Q_o$  it follows that  $Q^{iv}, Q'''_1, \dots, Q'''_{r-1}, Q'_p, Q_o$  appear in this order in  $T$ .



As  $t_i$  is not adjacent to  $t_p$  for  $i \neq p - 1$  then  $Q'_i \notin T[Q^{iv}, Q_o]$ . Observe that  $\{x_1, y_1, z_o, t_i\} \subset Q'_i \cap Q'_{i+1}$ ,  $\{x_1, y_1, z_o\} \subset Q'_i \cap Q_o$  but  $t_i \notin Q_o$  for  $i \neq p$  and  $\{x_1, y_1, z_o, t_p, t'_1\} \subset Q'_{p-1} \cap Q^{iv}$ ,  $\{x_1, y_1, z_o, t_p\} \subset Q'_{p-1} \cap Q_o$  and  $t_1 \notin Q_o$ . Thus  $Q'_1, \dots, Q'_{p-1}Q^{iv}, Q'''_1, \dots, Q'''_{r-1}, Q'_p, Q_o$  appear in this order in  $T$ .

On the other hand,  $Q' \notin T[Q'_1, Q_o]$  since  $z_o$  is not adjacent to  $t$ ;  $\{x_1, y_1, t_1\} \subset Q' \cap Q'_1$ ,  $\{x_1, y_1\} \subset Q' \cap Q_o$  but  $t_1 \notin Q_o$ , so  $Q', Q'_1, \dots, Q'_{p-1}, Q^{iv}, Q'''_1, \dots, Q'''_{r-1}, Q'_p, Q_o$  appear in this order in  $T$ . Since  $a$  is not adjacent to  $t_i$  for  $i = 1, \dots, p$  and  $a$  is not adjacent to  $x_2$ , then  $Q_a \notin T[Q', Q_o]$ . Vertex  $b$  is not adjacent to  $t_i$  for  $i = 1, \dots, p$  and is also not adjacent to  $y_1$ , so  $Q_b \notin T[Q', Q_o]$ . As  $x_1, y_1 \in (Q_a \cap Q') - (Q_o \cap Q_b)$  and  $x_2, y_2 \in (Q_o \cap Q_b) - (Q_a \cap Q')$ , then  $Q_a, Q', Q'_1, \dots, Q'_{p-1}, Q^{iv}, Q'''_1, \dots, Q'''_{r-1}, Q'_p, Q_o, Q_b$  appear in this order in  $T$ . Since  $x_1 \in Q_a \cap Q_o$  and  $x_2 \in Q' \cap Q_b$  it follows that  $T[Q_a, Q_b]$  is a directed path of  $T$ .

Using the same argument of Case 2c for Type 7, we have that  $Q_a, Q', \dots, Q_o, Q''_q, \dots, Q''_1, Q'', Q_{o-1}, \dots, Q$  appear in this order in  $T$ , and also  $Q_b \notin T[Q_a, Q]$ . □

We will say that there is a *special connection* between two non adjacent vertices  $a$  and  $b$  if

- (1) there exists a connection of Type 1 between  $a$  and  $b$ ; or
- (2) there exist two induced paths in  $G$ ,  $P = a, y_1, \dots, y_n, b$  and  $Q = a, x_1, \dots, x_m, b$ , such that
  - (a)  $P \cap Q = \{a, b\}$
  - (b) if  $\{x_i, x_{i+1}, y_j, y_{j+1}\}$  is a clique for  $i \in \{1, \dots, m - 1\}$  and  $j \in \{1, \dots, n - 1\}$  then there is a connection of Type  $k \in \{3, 4, 5, 6, 7, 8, 9\}$  between  $x_{i-1}$  and  $y_{j+2}$  for  $i \neq 1, j \neq n - 1$ , or between  $a$  and  $y_{j+2}$  for  $j \neq n - 1$ , or between  $x_{i-1}$  and  $b$  for  $i \neq 1$ , or between  $a$  and  $b$ , or between  $y_{j-1}$  and  $x_{i+2}$  for  $j \neq 1, i \neq m - 1$ , or between  $a$  and  $x_{i+2}$  for  $i \neq m - 1$  or between  $y_{j-1}$  and  $b$  for  $j \neq 1$ .
  - (c) if  $\{x_i, x_{i+1}, y_j, y_{j+1}\}$  is not a clique then there is a special connection of Type 2 between  $x_{i-1}$  and  $y_{j+2}$  for  $i \neq 1, j \neq n - 1$ , or between  $a$  and  $y_{j+2}$  for  $j \neq n - 1$ , or between  $x_{i-1}$  and  $b$  for  $i \neq 1$ , or between  $a$  and  $b$ , or between  $y_{j-1}$  and  $x_{i+2}$  for  $j \neq 1, i \neq m - 1$ , or between  $a$  and  $x_{i+2}$  for  $i \neq m - 1$ , or between  $y_{j-1}$  and  $b$  for  $j \neq 1$ .

#### 4. PROPERTIES OF MODELS THAT CANNOT BE ROOTED ON A BOLD MAXIMAL CLIQUE

If  $G$  is a  $DV$  graph that has a strong asteroidal triple  $a_1, a_2, a_3$ , then  $G \setminus N[a_i]$  is a connected graph for  $i = 1, 2, 3$ . Hence for every  $T$ , a  $DV$ -model of  $G$ ,  $N[a_i]$  for  $i = 1, 2, 3$  must be a leaf of  $T$ . Let  $C_i$  be the closest vertex to  $N[a_i]$  such that it has degree at least three, for  $i = 1, 2, 3$ . If  $|V(T[N[a_i], C_i])| > 2$ , we will denote by  $e_i = A_i B_i$  the edge in  $T[N[a_i], C_i]$ , with  $A_i$  the neighbor of  $N[a_i]$  and  $B_i \neq N[a_i]$ .

If there exists an edge dominated by  $e_i$  then we choose  $e'_i$  to be maximally farthest from  $e_i$ . We denote this edge by  $e'_i = A'_i B'_i$  with  $B'_i \in T[A'_i, C_i]$ .

We will say that a DV graph  $G$  is *minimally non rooted on a maximal clique  $H$*  if no DV-model  $T$  of  $G$  can be rooted on  $H$  but for every  $x \in V(G) \setminus H$ ,  $G \setminus x$  has a DV-model that can be rooted on  $H$ .

In what follows, if  $T$  has three leaves we will denote by  $C$  the vertex of degree exactly three in  $T$ .

**Lemma 1.** *Let  $G$  be a DV graph that has a strong asteroidal triple  $a_1, a_2, a_3$  and is minimal non rooted on  $N[a_3]$ .*

- (1) *Let  $T$  be a DV-model of  $G$ . Then:*
  - (a) *For all  $e$  edge in  $T[N[a_i], C_i]$  for  $i = 1, 2$ , there are at least two vertices  $x, y \in \text{lab}(e)$  such that  $T_x$  and  $T_y$  have different end towards  $C_i$ .*
  - (b) *There are not twin edges in  $T[N[a_i], C_i]$  for  $i = 1, 2, 3$ .*
  - (c) *If  $|V(T[N[a_i], C_i])| > 2$  for  $i = 1, 2$  then there is a dominated edge by  $e_i$  that is not in  $T[N[a_i], C_i]$ .*
- (2) *If  $T$  is a DV-model of  $G$  that has three leaves, then  $T$  does not have twin edges, one in  $T[N[a_i], C]$  for  $i = 1, 2$  and the other in  $T[N[a_3], C]$ .*

*Proof.* (1) (a) Suppose by contradiction that every vertex  $x$  in  $\text{lab}(e)$  has the same end to  $C_i$ . Let  $e = AB \in T[N[a_i], C_i]$  for  $i = 1, 2$  with  $B \in T[A, C_i]$  and  $T' = T - E(T[N[a_i], B])$ . All vertices of  $\text{lab}(e)$  are twins in  $G_{T'}$ . Let  $T''$  be a DV-model of  $G_{T'}$ . Since  $a_1, a_2, a_3$  is a strong asteroidal triple,  $N[a_3]$  and  $N[a_j]$  for  $j \neq i, 3$  are leaves of  $T''$ , and by minimality we have that  $T''$  can be rooted on  $N[a_3]$ . For  $x \in \text{lab}(e)$ ,  $T''_x = T''[Z, W]$  and  $W \in T''[Z, N[a_3]]$ . Let  $\bar{T} = T'' + ZA + T[A, N[a_i]]$ . It is easy to check that  $\bar{T}$  is a DV-model of  $G$  that can be rooted on  $N[a_3]$ , a contradiction.

(b) Suppose by contradiction that there are two twin edges in  $T[N[a_i], C_i]$  for  $i \in \{1, 2, 3\}$ . Let  $e = AB$  and  $e' = A'B'$  be twin edges with  $A, B, A', B'$  appearing in this order in  $T[N[a_i], C_i]$ , and  $T' = T - E(T[A, B']) + AB'$ . By minimality, there is  $T''$  a DV-model of  $G_{T'}$  that can be rooted on  $N[a_3]$ . Let  $\tilde{e} = \tilde{A}\tilde{B}'$  be an equivalent edge of  $AB'$  in  $T''$ . Thus, it is possible to build a DV-model of  $G$  from  $T''$  by adding  $T(A, B')$  as follows:  $T'' - \tilde{A}\tilde{B}' + \tilde{A}T(A, B)\tilde{B}'$ . Clearly, this DV-model can be rooted on  $N[a_3]$ , a contradiction.

(c) Suppose by contradiction that  $e_i$  cannot dominate an edge outside of  $T[N[a_i], C_i]$ , i.e.,  $e'_i \in T[N[a_i], C_i]$ . Let  $T' = T - T[N[a_i], A'_i]$ . By the choice of  $e'_i$ , it is clear that  $A'_i$  is always a leaf in every DV-model of  $G_{T'}$ . By minimality, there is  $T''$  a DV-model of  $G_{T'}$  that can be rooted on  $N[a_3]$ . It is easy to see that  $T'' + T[A'_i, N[a_i]]$  is a DV-model of  $G$  that can be rooted on  $N[a_3]$ , a contradiction.

(2) Suppose by contradiction that  $e$  and  $e'$  are twin edges, one in  $T[N[a_i], C]$  and the other in  $T[C, N[a_3]]$  for  $i = 1, 2$ . Let  $e = AB \in T[N[a_i], C]$  and  $e' = A'B' \in T[C, N[a_3]]$  with  $B \in T[A, C]$  and  $B' \in T[C, A']$ . Let

$T' = T - \{e, e'\} + AB' + BA'$ . It is a DV-model of  $G$ . Since  $T$  cannot be rooted on  $N[a_3]$ , there is a vertex crossing by  $C$  in  $T[N[a_1], N[a_2]]$ , and then  $B \neq C$ . As there is a vertex crossing by  $C$  in  $T[N[a_i], N[a_3]]$  then there is no vertex crossing by  $C$  in  $T[N[a_j], C]$  for  $j \neq i, 3$  because  $G$  is a DV graph. Hence  $T'$  can be rooted on  $N[a_3]$ , a contradiction.  $\square$

**Theorem 2.** *Let  $G$  be a DV graph that has a strong asteroidal triple  $a_1, a_2, a_3$  and is minimal non rooted on  $N[a_3]$ . If  $T$  is a DV-model of  $G$  that has three leaves, two twin edges one in  $T[N[a_1], C]$  and the other in  $T[N[a_2], C]$ , then there is a special connection of Type 1 or Type 2 between  $a_1$  and  $a_2$ .*

*Proof.* Since  $T$  has twin edges it follows that  $|V(T[N[a_i], C])| \geq 3$  for some  $i \in \{1, 2\}$ . If there exists a vertex  $x \in N[a_1] \cap N[a_2]$  then there is a special connection of Type 1 between  $a_1$  and  $a_2$ .

Suppose that there is not a vertex in this condition. By Lemma 1a, and by the position of twin edges in  $T$  it follows that  $N[a_1]A_1$  cannot be a dominated edge of  $e_2$  if it exists, and  $N[a_2]A_2$  cannot be a dominated edge of  $e_1$  if it exists.

Let  $e \in T[N[a_1], C]$  and  $e' \in T[C, N[a_2]]$  be twin edges such that their distance is maximum in  $T$ . As  $N[a_i]A_i$  is not dominated by  $e_j$  with  $\{i, j\} = \{1, 2\}$ , and by the choice of  $e'_i$ , it is clear that  $e'_1 \in T[e', A_2]$  and  $e'_2 \in T[e, A_1]$ . But by the election of  $e$  and  $e'$  to maximum distance in  $T$ ,  $e'_1$  must be  $e'$  and  $e'_2$  must be  $e$ . Then  $lab(e) = lab(e') \subset A_1 \cap A_2$ .

On the other hand, by Lemma 1a there are two vertices  $x, y \in lab(e)$  such that  $T_x = T[Ix, Dx]$ ,  $T_y = T[Iy, Dy]$ ,  $Ix \neq Iy$ ,  $Dx \neq Dy$  with  $Dx = N[a_2]$  and  $Iy = N[a_1]$ . Clearly,  $x \notin N[a_1]$  and  $y \notin N[a_2]$ . Also by Lemma 1a, there are vertices  $y_1 \in lab(N[a_1]A_1)$  and  $x_1 \in lab(N[a_2]A_2)$  such that  $Dy_1 \neq Dy = A_2$  and  $Ix_1 \neq Ix = A_1$ . Clearly,  $x_1 \notin N[a_1]$  and  $y_1 \notin N[a_2]$ . Observe that  $y_1 \notin lab(e'_2)$  and  $x_1 \notin lab(e'_1)$ . Hence, there is a special connection of Type 2 between  $a_1$  and  $a_2$ . More clearly,  $P = a_1, y, x_1, a_2$  and  $Q = a_1, y_1, x, a_2$  are paths in  $G$ , and  $\{x, y, y_1\}$ ,  $\{y, x, x_1\}$  are cliques of  $G$ .  $\square$

**Corollary 1.** *Let  $G$  be a DV graph that has a strong asteroidal triple  $a_1, a_2, a_3$  and is minimal non rooted on  $N[a_3]$ . If there exists  $T$  a DV-model of  $G$  with three leaves such that  $e'_i$  is in  $T[N[a_j], C]\{i, j\} = \{1, 2\}$ , then there is a special connection of Type 1 or Type 2 between  $a_1$  and  $a_2$ .*

*Proof.* If  $e'_1 = N[a_2]A_2$  or  $e'_2 = N[a_1]A_1$ , by Lemma 1a there is  $x \in N[a_1] \cap N[a_2]$ . Hence, there is a special connection of Type 1 between  $a_1$  and  $a_2$ . Otherwise, as  $e'_1 \in T[A_2, C]$  and  $e'_2 \in T[A_1, C]$  then  $lab(e'_1) = lab(e'_2)$ . So by Theorem 2 there is a special connection of Type 1 or Type 2 between  $a_1$  and  $a_2$ .  $\square$

**Claim 1.** *Let  $G$  be a DV graph with a strong asteroidal triple  $a_1, a_2, a_3$  and such that no DV-model of  $G$  can be rooted on  $N[a_3]$ . If there exists  $T$  a DV-model of  $G$  with three leaves, and  $e = AB \in T[N[a_3], C]$  is dominated by  $e' \in T(N[a_1], C]$  with  $B \in T[C, A]$ , then no edge of  $T(e', C]$  can have in its label vertices with the same end towards  $N[a_1]$ .*

By way of contradiction, suppose that there is  $e'' \in T(e', C]$  such that all vertices in  $lab(e'')$  have the same end towards  $N[a_1]$ . Let  $A_1$  be the end of these vertices. Let  $e'' = A''B''$  with  $B'' \in T[A'', C]$ . As  $lab(e) \subset lab(e')$  and  $e'' \in T(e', C]$  then  $lab(e) \subset lab(e'') \subset A_1$ . Also, the vertices in  $lab(e'')$  have  $A_1$  as a leaf. Let  $T' = T - \{e'', e\} + B''A_1 + AA''$ . It is clear that  $T'$  is a DV-model of  $G$ . Observe that there is no vertex crossing by  $A_1$  in  $T'[N[a_1], C]$  because  $A_1$  is a leaf of each vertex in  $lab(e'')$ . Also, there is no vertex crossing by  $C$  in  $T'[N[a_2], B]$  since there is no vertex crossing by  $C$  in  $T[N[a_2], N[a_3]]$ . Clearly,  $T'$  is a DV-model of  $G$  that can be rooted on  $N[a_3]$ , a contradiction. This proves Claim 1.

**Claim 2.** *Let  $G$  be a DV graph with a strong asteroidal triple  $a_1, a_2, a_3$  and such that no DV-model of  $G$  can be rooted on  $N[a_3]$ . Let  $T$  be a DV-model of  $G$  with three leaves, and let  $X, Y \in V(T)$  be such that one and only one of them is in  $T[N[a_i], C]$  for  $i = 1, 2$  and the other is in  $T[N[a_3], C]$ . If  $e = AB$  is an edge in  $T[X, C]$  with  $B \in T[A, C]$ , which is dominated by  $D \in T[Y, C]$ , then  $\forall e' \in T[C, D']$ ,  $lab(e') \not\subseteq B$  whenever  $DD' \in E(T[Y, C])$ .*

By way of contradiction, suppose that there is  $e' \in T[C, D']$  such that  $lab(e') \subset B$ . Let  $e' = A''B''$  be such that  $B'' \in T[C, A'']$ . Clearly  $T' = T - \{e, e'\} + A''B + AB''$  is a model of  $G$ .

Suppose that  $X$  is in  $T[N[a_1], C]$ . Observe that  $e' \in T[C, D']$ , and  $A''$  may be  $D'$ . Since  $e$  is dominated by  $D$  then  $lab(e) \subset B''$ . Thus  $T'$  is a DV-model of  $G$ . Clearly,  $T'$  does not have a vertex crossing by  $C$  in  $T'[N[a_1], N[a_2]]$  since there is no vertex crossing by  $C$  in  $T[N[a_3], N[a_2]]$ . Hence,  $T'$  can be rooted on  $N[a_3]$ , a contradiction.

The proof is the same if  $Y$  is in  $T[N[a_1], C]$ . This proves Claim 2.

**Election 1 of vertices in label of edges:**

Let  $T$  be a DV-model of  $G$  and  $A, B$  be vertices that appear in this order in  $T$ . Let  $e(1)$  be the edge in  $T[A, B]$  incident to  $A$ .

- Take a vertex  $w_1 \in lab(e(1))$  such that  $T_{w_1}$  is the shortest towards  $B$ . Let  $T_{w_1} = T[Iw_1, Dw_1]$  with  $A \in T[Iw_1, Dw_1]$ . If  $B \notin T[A, Dw_1]$  then we repeat the following process,  $i > 0$ :
- Let  $e(i+1)$  be the edge in  $T[Dw_i, B]$  incident to  $Dw_i$  and  $w_{i+1} \in lab(e(i+1))$  such that  $T_{w_{i+1}}$  is the shortest towards  $B$ , if  $w_{i+1} \in Iw_i$  (for  $i = 1$ , take  $A$  instead of  $Iw_i$ ) then  $w_i = w_{i+1}$ , we continue until cover all  $T[A, B]$ .

Observe that  $T_{w_i} \not\subseteq T_{w_{i+1}}$ .

The preceding election of vertices is a technical tool in order to define special connections of Type 4, 5, ..., or 9.

5. PROOF OF THE MAIN THEOREM

Finally, in this section we give the result that is the goal of this article, a characterization of rooted directed path graphs whose rooted models cannot be rooted on a bold maximal clique.

**Theorem 3.** *Let  $G$  be a DV graph with a strong asteroidal triple  $a_1, a_2, a_3$  and leafage three. There is a special connection between  $a_1$  and  $a_2$  if and only if no DV-model of  $G$  can be rooted on  $N[a_3]$ .*

*Proof.*  $\Rightarrow$  By Theorem 1.

$\Leftarrow$  Suppose that  $G$  is the smallest graph such that no DV-model of  $G$  can be rooted on  $N[a_3]$ . Since  $l(G) = 3$  and  $G \setminus N[a_i]$  is a connected graph for  $i = 1, 2, 3$  then  $N[a_i]$  is a leaf in every model of  $G$ . Let  $T$  be a DV-model of  $G$  that reaches the leafage, and  $C$  be the vertex of degree three in  $T$ . Since  $T$  cannot be rooted on  $N[a_3]$  then there is a vertex crossing by  $C$  in  $T[N[a_1], N[a_2]]$ . By Lemma 1b, we can assume that there are not two edges with the same label in  $T[N[a_i], C]$  for all  $i \in \{1, 2, 3\}$ . Clearly if  $T[N[a_i], C]$  has exactly two vertices for  $i = 1, 2$  then there exists a vertex  $x \in N[a_1] \cap N[a_2]$ , so there is a special connection of Type 1 between  $a_1$  and  $a_2$ .

Now, we can assume that  $T[N[a_i], C]$ , for some  $i \in \{1, 2\}$ , has at least three vertices.

Suppose that  $T[N[a_1], C]$  has at least three vertices. Thus there exists  $e_1 \in T[N[a_1], C]$ , and by Lemma 1c there exists  $e'_1 \notin T[N[a_1], C]$ . If  $e'_1$  is in  $T[N[a_2], C]$  then by Corollary 1 there is a special connection of Type 1 or 2 between  $a_1$  and  $a_2$ .

Suppose that it is in  $T[N[a_3], C]$ . As  $T$  is a DV-model of  $G$ , and there is a vertex crossing by  $C$  in  $T[N[a_1], N[a_2]]$  if  $CN[a_2] \notin E(T)$  then  $e'_2 = N[a_1]A_1$ . Therefore, by Corollary 1, there is a special connection of Type 1 or Type 2 between  $a_1$  and  $a_2$ .

Consider  $CN[a_2] \in E(T)$ .

By Lemma 1a,  $|lab(N[a_1]A_1)|$  and  $|lab(N[a_2]C)|$  are greater than one. Let  $x_1, y_1 \in lab(N[a_1]A_1)$  and  $x_2, y_2 \in lab(N[a_2]C)$  be such that  $|\{Q \in C(G) : x_i \in Q\}| > |\{Q \in C(G) : y_i \in Q\}| > 1$ ,  $|\{Q \in C(G) : x_i \in Q\}|$  is maximum, and  $|\{Q \in C(G) : y_i \in Q\}|$  is minimum for  $i = 1, 2$ . Observe that if  $x_1 \in N[a_2]$  or  $x_2 \in N[a_1]$  then there is a special connection of Type 1 between  $a_1$  and  $a_2$ .

In what follows, we suppose that  $x_1 \notin N[a_2]$  and  $x_2 \notin N[a_1]$ . Let  $X_i$  be the leaf of  $T_{x_i}$  and  $Y_i$  be the leaf of  $T_{y_i}$  different from  $N[a_i]$  for  $i = 1, 2$  respectively. Observe that  $X_2, Y_2 \in T[N[a_1], N[a_2]]$  but  $X_1, Y_1$  may be in  $T[C, N[a_3]]$ .

First of all, we know that  $lab(e'_1) \subset A_1$ . As  $lab(e'_1) \not\subseteq N[a_1]$ , since  $G \setminus N[a_1]$  is a connected graph, there is a vertex  $v \in lab(e'_1) \cap A_1 - N[a_1]$ .

In what follows, we will analyze several cases taking into account if  $\{x_1, x_2, y_1, y_2\}$  is a clique of  $G$  or not. We will study two situations depending on whether there is an edge  $e \in T[N[a_1], X_2]$  such that  $lab(e) \subset C$ .

**Case 0:**  $T_{x_1} \cap T_{x_2} \neq \emptyset$  but  $T_{y_2} \cap T_{x_1} = \emptyset$  and  $T_{y_1} \cap T_{x_2} = \emptyset$ . Clearly  $x_1 \notin C$ . Let  $P = a_1, y_1, v, y_2, a_2$  and  $Q = a_1, x_1, x_2, a_2$  be paths in  $G$ . Then there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{y_1, v, x_1\}$ ,  $\{v, x_1, x_2\}$ ,  $\{v, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, y_2$  and between  $y_1, a_2$ ; see Figure 2.

- There is not an edge  $e \in T[N[a_1], X_2]$  such that  $lab(e) \subseteq C$ .

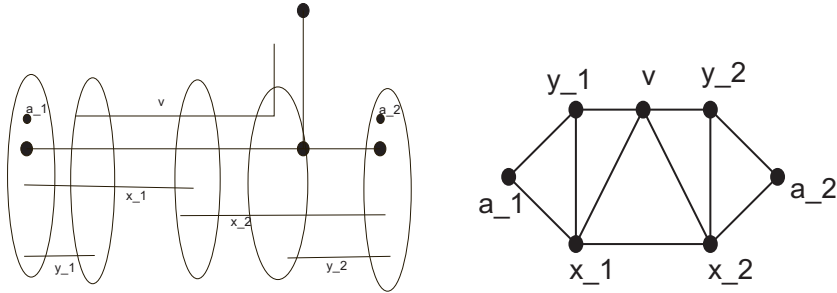


FIGURE 2. Case 0: Type 2 between  $a_1, y_2$  and  $y_1, a_2$ .

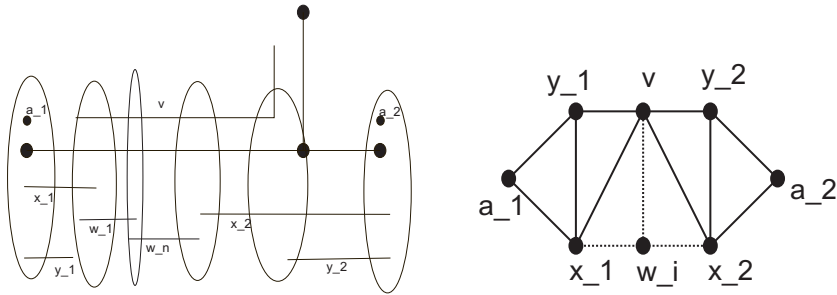


FIGURE 3. Case 1.1: Type 2 between  $a_1, w_2; w_{n-1}, a_2$  and Type 1 between  $w_i, w_{i+2}$ .

**Case 1:**  $T_{x_1} \cap T_{x_2} = \emptyset$ . By the choice of  $x_1$ , each vertex in  $lab(e'_1)$  must have  $A_1$  as a leaf. Clearly, there is a path  $P = a_1, y_1, v, y_2, a_2$  in  $G$  between  $a_1$  and  $a_2$ . We will need another path  $Q$  in  $G$  between  $a_1$  and  $a_2$ . Observe that: a) by the election of  $x_1$ , for all  $\bar{e} \in T[X_1, X_2]lab(\bar{e}) \cap N[a_1] = \emptyset$ ; b) by Claim 1, as  $e'_1 \in T[C, N[a_3]]$  is a dominated edge by  $e_1 \in T(N[a_1], C]$ ,  $\forall \bar{e} \in T[X_1, X_2]$ , the vertices in its label cannot have the same end to  $N[a_1]$ . Since  $v \in lab(\bar{e})$  and its end is  $A_1$ , it follows that there is  $w \in lab(\bar{e}) - A_1$ .

As was mentioned above, we need to search another path and Claim 1 will provide its vertices. We choose vertices through Election 1, we take  $A = X_1, B = X_2$  and  $w_i \notin A_1$  for  $i = 1, \dots, n$ . Clearly,  $Q = a_1, x_1, w_1, \dots, w_n, x_2, a_2$  is a path in  $G$  different from  $P$  between  $a_1$  and  $a_2$ . Observe that  $w_n$  may be in  $C$ . In this last case, as  $lab(e) \not\subseteq C$  for all  $e \in T[C_{a_1}, X_2]$  it follows that there exists a vertex  $w' \in lab(e(n)) - C$  that has  $A_1$  as one of its leaves. Recall that  $w_n$  was chosen in the label of  $e(n)$ .

Next, we will study if there is a clique in  $G$  of size four with two vertices of  $P$  and two vertices of  $Q$ .

**Case 1.1:** There is not a clique in  $G$  of size four with two vertices of  $P$  and two vertices of  $Q$ . Then  $y_1$  and  $w_1$  are not adjacent vertices; also

$y_2$  and  $w_n$  are not adjacent vertices. Therefore, there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{y_1, v, x_1\}$ ,  $\{x_1, v, w_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$ ,  $\{w_n, v, x_2\}$ ,  $\{v, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$  and  $w_{n-1}, a_2$ , and of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ ; see Figure 3.

**Case 1.2:** There is only one clique in  $G$  of size four with two vertices of  $P$  and two vertices of  $Q$ . First, suppose that  $\{x_1, y_1, w_1, v\}$  is the clique. Since there is one and only one clique of size four,  $y_2$  and  $w_n$  are not adjacent vertices. On the other hand,  $y_1$  and  $w_1$  are adjacent vertices and  $w_1 \notin A_1$ , then  $A_1$  must be separated by a vertex  $s_1$  in direction to  $N[a_1]$ . By the choice of  $y_1$ ,  $s_1 \notin N[a_1]$  so it is a simplicial vertex of  $G$ . Let  $s_2$  be a separator vertex of  $X_1$  to  $C$  such that  $|\{Q \in C(G) : s_2 \in Q\}|$  is minimum. By the election of  $w_1$ , it is clear that  $s_2 \notin Dw_1$ , then it is not adjacent vertex to  $w_2$ . Therefore there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$ ,  $\{w_n, v, x_2\}$ ,  $\{v, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ , and  $\{x_1, y_1, w_1, v, s_1, s_2\}$  induces an antenna.

Observe that there is a special connection of Type 3 between  $a_1$  and  $w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ , and of Type 2 between  $w_{n-1}, a_2$ ; see Figure 4.

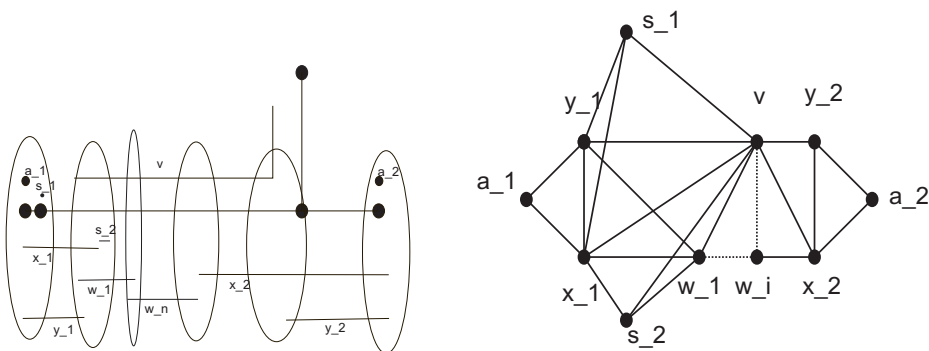


FIGURE 4. Case 1.2: Type 3 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; Type 2 between  $w_{n-1}, a_2$ .

**Case 1.3:** There is one only one clique in  $G$  of size four with two vertices of  $P$  and two vertices of  $Q$ . Now, suppose that  $\{x_2, y_2, w_n, v\}$  is the clique. As there is only one clique of size four,  $y_1$  and  $w_1$  are not adjacent vertices. We will analyze two situations depending on whether  $w_n$  is in  $C$ .

- (1)  $w_n \notin C$ . Let  $s_3$  be a separator vertex of  $C$  to  $N[a_3]$ . Clearly  $s_3$  is not adjacent to  $w_n$ . Let  $s_4$  be a separator vertex of  $X_2$  to  $N[a_1]$  such that  $|\{Q \in C(G) : s_4 \in Q\}|$  is minimum. By the election of  $w_n$ ,  $s_4$  is not

adjacent vertex to  $w_{n-1}$ . Then there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{y_1, v, x_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ , and  $\{x_2, y_2, w_n, v, s_3, s_4\}$  induces an antenna.

Observe that there is a special connection of Type 2 between  $a_1$ ,  $w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n-2\}$ , and of Type 3 between  $w_{n-1}, a_2$ ; see Figure 5.

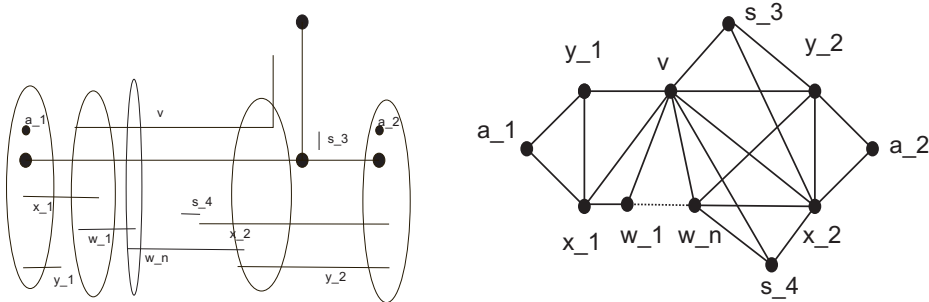


FIGURE 5. Case 1.3.1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; Type 3 between  $w_{n-1}, a_2$ .

- (2)  $w_n \in C$ . Hence there is a vertex in  $lab(e(n)) - C$  that has  $A_1$  as one of its leaves. Let  $w'$  be the vertex such that  $|\{Q \in C(G) : w' \in Q\}|$  is maximum. Let  $W'$  be the other leaf of  $w'$ .

If  $w' \in X_2$  then we take  $P = a_1, y_1, w', x_2, a_2$  and  $Q = a_1, x_1, w_1, \dots, w_n, y_2, a_2$  paths in  $G$ .

In case that  $y_2 \notin W'$  then there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n-2\}$ , and of Type 2 between  $w_{n-1}, a_2$ ; see Figure 6.

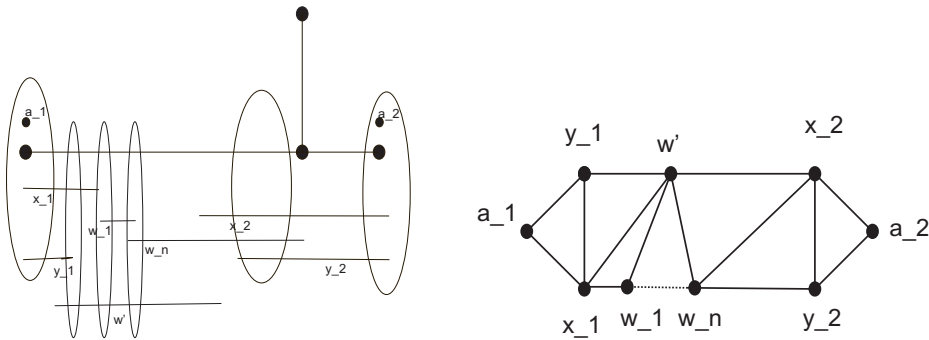


FIGURE 6. Case 1.3.2: Type 2 between  $a_1, w_2$  and  $w_{n-1}, a_2$ ; Type 1 between  $w_i, w_{i+2}$ .



In case that  $y_2 \in W'$ , by the same argument used in 1 taking  $w_n, w'$  instead of  $w_n, v$ , there is a special connection of Type 3 between  $w_{n-1}, a_2$ ; see Figure 7.

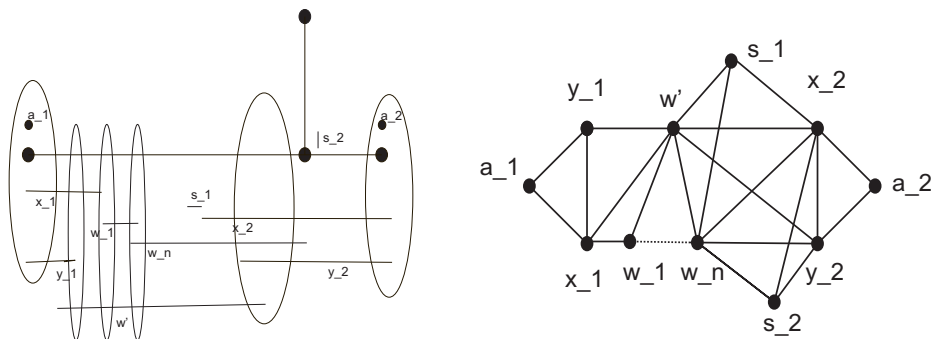


FIGURE 7. Case 1.3.2: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$  and Type 3 between  $w_{n-1}, a_2$ .

Now, suppose that  $w' \notin X_2$ . Then we choose vertices in label of edges of  $T[W', X_2]$  that are not in  $C$  through Election 1 with  $A = W'$  and  $B = X_2$ . Let  $t_i$  be these vertices for  $i = 1, \dots, m$  such that  $t_1$  is the first vertex chosen. Observe that by the choice of  $w', t_1 \notin A_1$  and by the election of  $w_n, t_1 \notin \text{lab}(e(n))$ , then  $t_1$  is not adjacent to  $w_{n-1}$ . Let  $P = a_1, y_1, w', t_1, \dots, t_m, x_2, a_2$  and  $Q = a_1, x_1, w_1, \dots, w_n, y_2, a_2$  be paths in  $G$ . Note that  $\{w_n, t_m, y_2, x_2\}$  and  $\{y_1, x_1, w', w_1\}$  may be cliques. Clearly,  $\{y_1, x_1, w', w_1\}$  is not a clique because  $\{y_1, x_1, v, w_1\}$  is not a clique. In case that  $\{w_n, t_m, x_2, y_2\}$  is not a clique then there is a special connection between  $a_1$  and  $a_2$ . More clearly  $\{a_1, y_1, x_1\}, \{w', x_1, w_1\}, \{w', w_i, w_{i+1}\}_{i=1, \dots, n-1}, \{w', t_1, w_n\}, \{t_i, t_{i+1}, w_n\}_{i=1, \dots, m}, \{t_m, x_2, w_n\}, \{x_2, w_n, y_2\}, \{x_2, y_2, a_2\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w', t_2$  and between  $t_i, t_{i+2}$  with  $i \in \{1, \dots, m-2\}$ , and of Type 2 between  $t_{m-1}, a_2$ ; see Figure 8.

In case that  $\{w_n, t_m, y_2, x_2\}$  is a clique, let  $s_3$  be a separator vertex of  $C$  to  $N[a_3]$ . Clearly,  $s_3$  is not adjacent to  $t_m$  since  $t_m \notin C$ . Let  $s_4$  be a separator vertex of  $X_2$  to  $N[a_1]$  such that  $|\{Q \in C(G) : s_4 \in Q\}|$  is minimum. By the election of  $t_m, s_4$  is not adjacent vertex to  $t_{m-1}$ . Hence, there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, y_1, x_1\}, \{w', x_1, w_1\}, \{w', w_i, w_{i+1}\}_{i=1, \dots, n-1}, \{w', t_1, w_n\}, \{t_i, t_{i+1}, w_n\}_{i=1, \dots, m}, \{x_2, y_2, a_2\}$  are cliques of  $G$ , and  $\{t_m, x_2, y_2, w_n, s_3, s_4\}$  induces an antenna.

Observe that there is a special connection of Type 3 between  $t_{m-1}$  and  $a_2$ ; see Figure 9.

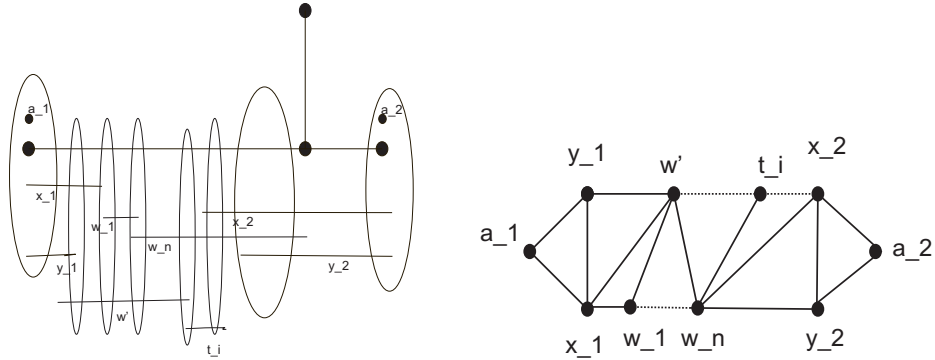


FIGURE 8. Case 1.3.2: Type 2 between  $a_1, w_2$  and  $t_{m-1}, a_2$ ; Type 1 between  $w', t_2$  and  $t_i, t_{i+2}$ .

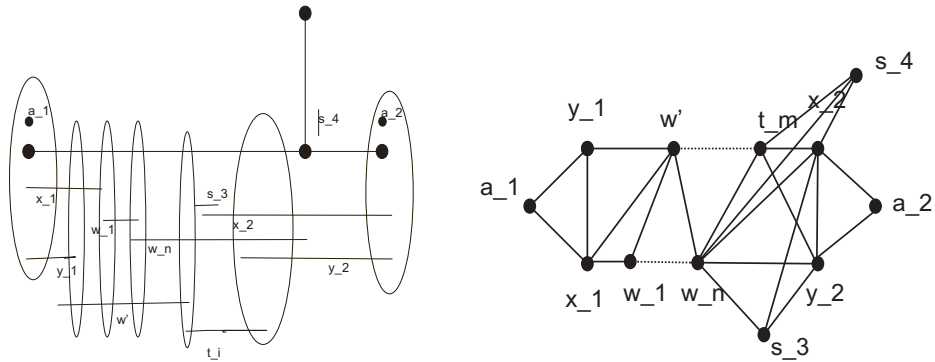


FIGURE 9. Case 1.3.2: Type 2 between  $a_1, w_2$ ; Type 1 between  $w', t_2$  and  $t_i, t_{i+2}$ ; Type 3 between  $t_{m-1}, a_2$ .

**Case 1.4:** There are two cliques in  $G$  of size four with two vertices of  $P$  and two vertices of  $Q$ , and they are  $\{x_1, y_1, w_1, v\}$  and  $\{x_2, y_2, w_n, v\}$ . In this situation, we obtain a combination of the previous cases.

**Case 2:**  $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$  but  $T_{y_2} \cap T_{x_1} = \emptyset$ , or  $T_{x_1} \cap T_{x_2} \cap T_{y_2} \neq \emptyset$  but  $T_{y_1} \cap T_{x_2} = \emptyset$ . In both situations it is clear that there are two paths  $P = a_1, y_1, v, y_2, a_2$  and  $Q = a_1, x_1, x_2, a_2$  in  $G$  between  $a_1$  and  $a_2$ . Next, we will study whether there is a clique in  $G$  of size four with two vertices of  $P$  and two vertices of  $Q$ .

**Case 2.1:**  $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$  but  $T_{y_2} \cap T_{x_1} = \emptyset$ . Clearly  $\{x_1, y_1, v, x_2\}$  is a clique of  $G$ . By the election of  $x_1$ , every vertex of  $lab(e'_1)$  has  $A_1$  as a leaf. It is necessary to study two situations depending on whether  $T_{x_2}$  and  $T_v$  have the same end to  $N[a_1]$ —or, more clearly, on whether  $A_1$  is a leaf of both of them.

First, we suppose that  $T_{x_2}$  and  $T_v$  have the same leaf in direction to  $N[a_1]$ , i.e., both of them have  $A_1$  as a leaf. By Claim 1, for each  $\bar{e} \in T[X_1, Y_2]$  the vertices in its label cannot have the same end to  $N[a_1]$ . As  $T_v$  and  $T_{x_2}$  have the same leaf  $A_1$  to  $N[a_1]$ , we choose vertices  $w_i \in \text{lab}(\bar{e}) - A_1$  through Election 1 with  $i = 1, \dots, n$ . Let  $P' = a_1, y_1, x_2, a_2$  and  $Q' = a_1, x_1, w_1, \dots, w_n, y_2, a_2$  be paths in  $G$  between  $a_1$  and  $a_2$ . Observe that  $\{x_1, x_2, y_1, w_1\}$  may be a clique. In case that  $w_1 \notin Y_1$ , there is not a clique of size four with two vertices of each path. Hence, there is a special connection between  $a_1, a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{x_1, x_2, y_1\}$ ,  $\{x_1, x_2, w_1\}$ ,  $\{w_i, w_{i+1}, x_2\}_{i=1, \dots, n-1}$ ,  $\{w_n, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Note that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ , and between  $w_n, a_2$ ; see Figure 10.

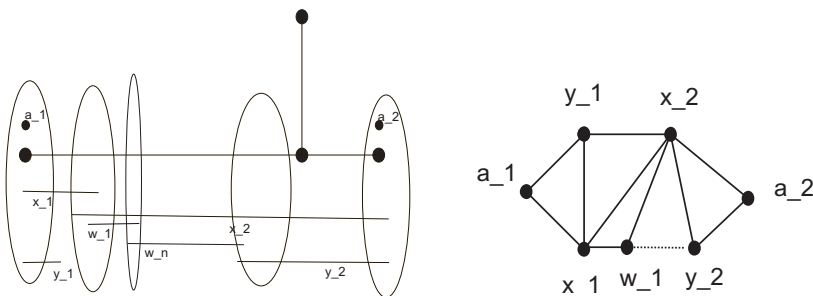


FIGURE 10. Case 2.1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$  and  $w_n, a_2$ .

In case that  $w_1 \in Y_1$ , as  $w_1 \notin A_1$  then  $Y_1 \neq A_1$ . Let  $s_1$  be a separator vertex of  $X_2 = A_1$  to  $N[a_1]$  such that  $|\{Q \in C(G) : s_1 \in Q\}|$  is minimum. By the choice of  $y_1$ ,  $s_1$  is a simplicial vertex of  $G$ . As  $w_1 \in Y_1$  then  $X_1 \neq Dw_1$ . Let  $s_2$  be a separator vertex of  $X_1$  to  $C_2$  such that  $|\{Q \in C(G) : s_2 \in Q\}|$  is minimum. By the election of  $w_1$ ,  $s_2$  is not adjacent to  $w_2$ . Hence there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{w_i, w_{i+1}, x_2\}_{i=1, \dots, n}$ ,  $\{w_n, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ , and  $\{x_1, y_1, w_1, x_2, s_1, s_2\}$  induces an antenna.

Note that there is a special connection of Type 3 between  $a_1, w_2$ ; of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ , and between  $w_n, a_2$ ; see Figure 11.

Finally, we can assume that  $T_{x_2}$  and  $T_v$  do not have the same leaf in direction to  $N[a_1]$ . By the election of  $x_1$ , it is clear that  $v \notin N[a_1]$ . As  $x_2 \notin N[a_1]$  and  $X_2 \neq A_1$ , we have that  $x_2 \notin A_1$ . Observe that  $Y_1 \neq A_1$  since  $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$ . Let  $s_1$  and  $s_2$  be vertices such that  $s_1$  is a separator vertex of  $A_1$  to  $N[a_1]$ ,  $s_2$  is a separator vertex of  $X_1$  to  $C$  and  $|\{Q \in C(G) : s_2 \in Q\}|$  is minimum. By the choice of  $y_1$ ,  $s_1 \notin N[a_1]$  so it is a simplicial vertex of  $G$ . If  $s_2$  is adjacent to  $y_2$ , let  $P' = a_1, y_1, x_2, a_2$

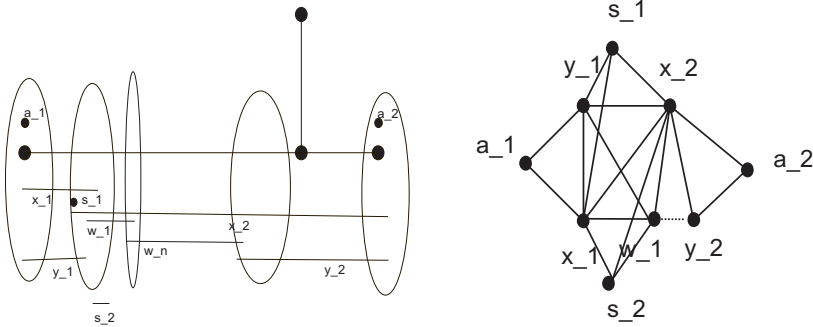


FIGURE 11. Case 2.1: Type 3 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$  and  $w_n, a_2$ .

and  $Q' = a_1, x_1, s_2, y_2, a_2$  be paths in  $G$  between  $a_1$  and  $a_2$ . Clearly, there is not a clique of size four with two vertices of each path. Then there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{x_1, x_2, s_2\}$ ,  $\{x_1, x_2, y_1\}$ ,  $\{s_2, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Note that there is a special connection of Type 2 between  $a_1, y_2$ , and of Type 1 between  $s_2, a_2$ ; see Figure 12.

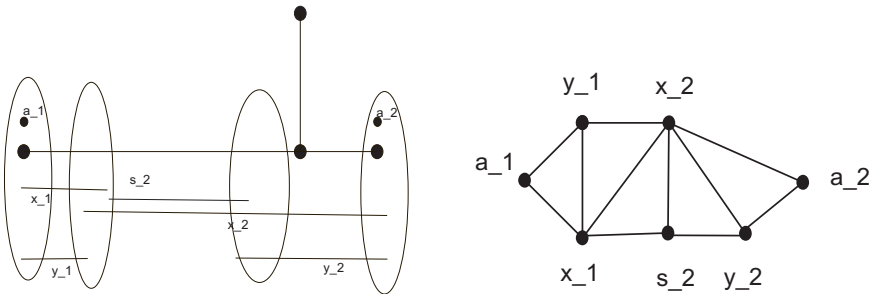


FIGURE 12. Case 2.1: Type 3 between  $a_1, y_2$ ; Type 1 between  $v, a_2$ .

If  $s_2$  is not adjacent to  $y_2$ , let  $P = a_1, y_1, v, y_2, a_2$  and  $Q = a_1, x_1, x_2, a_2$  be paths in  $G$  between  $a_1$  and  $a_2$ . Clearly there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{y_1, x_1, v, x_2, s_1, s_2\}$  induces an antenna and  $\{a_1, x_1, y_1\}$ ,  $\{v, x_2, y_2\}$ ,  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Note that there is a special connection of Type 3 between  $a_1, y_2$ , and of Type 1 between  $v, a_2$ ; see Figure 13.

**Case 2.2:**  $T_{x_1} \cap T_{x_2} \cap T_{y_2} \neq \emptyset$  but  $T_{y_1} \cap T_{x_2} = \emptyset$ . Clearly, there are two paths in  $G$  between  $a_1$  and  $a_2$ :  $P = a_1, x_1, x_2, a_2$  and  $Q = a_1, y_1, v, y_2, a_2$ . On the other hand,  $\{x_1, x_2, y_2, v\}$  is a clique of  $G$ . We will analyze if  $x_1$  is or is not in  $C$ .

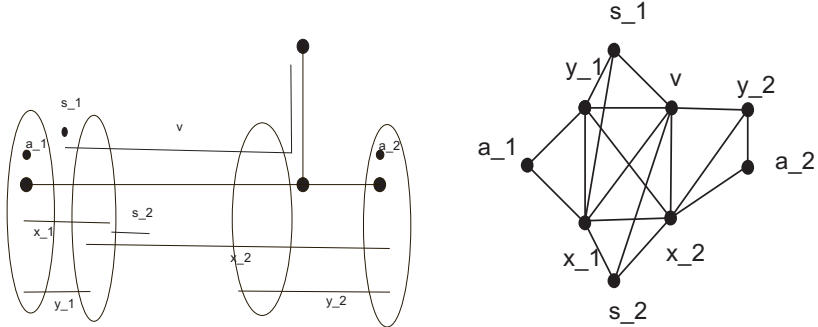


FIGURE 13. Case 2.1: Type 2 between  $a_1, y_2$ ; Type 1 between  $s_2, a_2$ .

First,  $x_1 \notin C$ . Clearly  $Y_2 \neq C$ . By the election of  $x_1$ , every vertex in  $lab(e_1)$  has  $A_1$  as a leaf. Let  $s_2$  be a separator vertex of  $X_2$  to  $N[a_1]$  such that  $|\{Q \in C(G) : s_2 \in Q\}|$  is minimum.

If  $s_2 \in Y_1$  then we can change the paths in order to make Type 2 appear. Let  $P' = a_1, y_1, s_2, x_2, a_2$  and  $Q' = a_1, x_1, y_2, a_2$  be paths in  $G$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{s_2, x_1, y_1\}$ ,  $\{s_2, x_1, x_2\}$ ,  $\{x_1, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Note that there is a special connection of Type 1 between  $a_1, s_2$ , and of Type 2 between  $y_1, a_2$ ; see Figure 14.

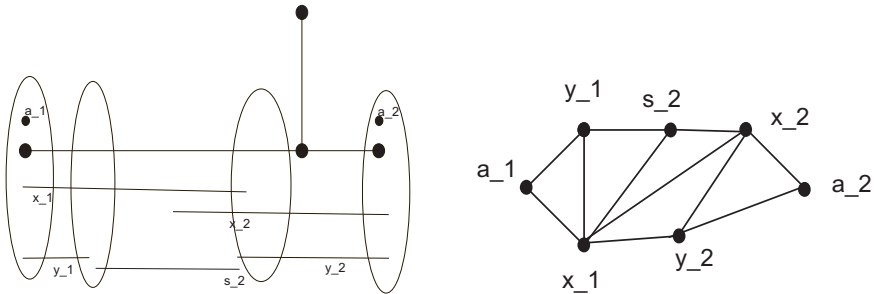


FIGURE 14. Case 2.2: Type 1 between  $a_1, s_2$ ; Type 2 between  $y_1, a_2$ .

If  $s_2 \notin Y_1$ , let  $s_1$  be a separator vertex of  $C$  to  $N[a_3]$ ; then there is a special connection between  $a_1, a_2$ . More clearly,  $\{y_2, x_2, v, x_1, s_1, s_2\}$  induces an antenna and  $\{a_1, x_1, y_1\}$ ,  $\{v, x_1, y_1\}$ ,  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 1 between  $a_1, v$ , and of Type 3 between  $y_1, a_2$ ; see Figure 15.

Finally,  $x_1 \in C$ . We know that  $\forall e \in T[Y_1, X_2], lab(e) \not\subseteq C$ . We choose  $w_i \in lab(e) - C$  through Election 1 with  $A = Y_1$  and  $B = X_2$ . Let  $P' = a_1, x_1, y_2, a_2$  and  $Q' = y_1, w_1, \dots, w_n, x_2, a_2$  be paths in  $G$  between  $a_1$  and  $a_2$ .

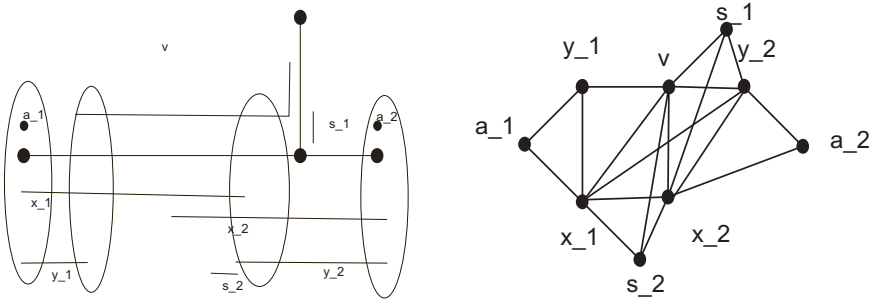


FIGURE 15. Case 2.2: Type 1 between  $a_1, v$ ; Type 3 between  $y_1, a_2$ .

If  $w_n$  is not adjacent to  $y_2$  then there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{x_1, y_1, w_1\}$ ,  $\{w_i, w_{i+1}, x_1\}_{i=1, \dots, n-1}$ ,  $\{w_n, x_1, x_2\}$ ,  $\{x_1, x_2, y_2\}$  and  $\{x_2, y_2, a_2\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 1 between  $a_1, w_1$ , between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n-2\}$ , and of Type 2 between  $w_{n-1}, a_2$ ; see Figure 16.

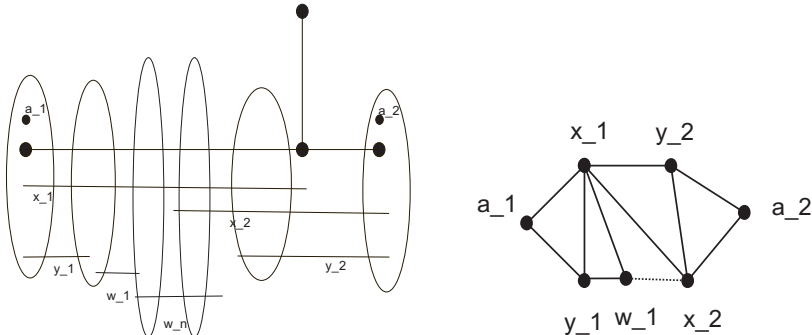


FIGURE 16. Case 2.2: Type 1 between  $a_1, w_1$ ; between  $w_i, w_{i+2}$ ; Type 2 between  $w_{n-1}, a_2$ .

If  $w_n$  is adjacent to  $y_2$  then  $Y_2 \neq C$  by the election of  $w_n \notin C$ . Clearly, there is a clique of size four with two of each path, it is  $\{x_1, x_2, y_2, w_n\}$ . Let  $s_1$  and  $s_2$  be vertices such that  $s_1$  is a separator of  $C$  to  $N[a_3]$ ,  $s_2$  is a separator of  $X_2$  to  $N[a_1]$  and  $|\{Q \in C(G) : s_i \in Q\}|$  is minimum for  $i = 1, 2$ . By the election of  $w_i$ ,  $s_2$  is not adjacent to  $w_{n-1}$ . Hence there is a special connection between  $a_1$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{x_1, y_1, w_1\}$ ,  $\{w_i, w_{i+1}, x_1\}_{i=1, \dots, n-1}$ ,  $\{x_2, y_2, a_2\}$  are cliques of  $G$ , and  $\{w_n, x_1, x_2, y_2, s_1, s_2\}$  induces an antenna.

Observe that there is a special connection of Type 1 between  $a_1, w_1$ , between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n-2\}$ , and of Type 3 between  $w_{n-1}, a_2$ ; see Figure 17.

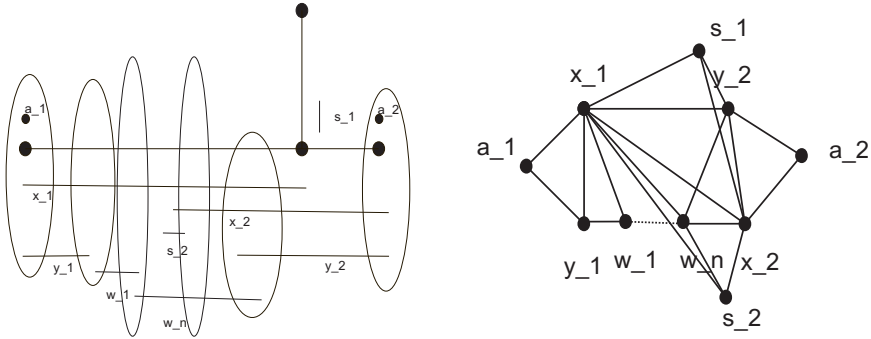


FIGURE 17. Case 2.2: Type 1 between  $a_1, w_1$  and  $w_i, w_{i+2}$ ; Type 3 between  $w_{n-1}, a_2$ .

**Case 3:**  $T_{x_1} \cap T_{x_2} \cap T_{y_1} \cap T_{y_2} \neq \emptyset$ . Clearly, there are two paths in  $G$  between  $a_1$  and  $a_2$ . Let  $P = a_1, y_1, y_2, a_2$  and  $Q = a_1, x_1, x_2, a_2$  be these paths. Also  $\{x_1, y_1, x_2, y_2\}$  is a clique of  $G$  which has two vertices of  $P$  and two vertices of  $Q$ . We will study two situations depending on whether  $x_1$  is in  $C$ .

First,  $x_1 \notin C$ . Since  $X_2 \neq N[a_1]$  then  $Y_1 \neq A_1$ . Let  $s_1$  be a separator vertex of  $X_2$  to  $N[a_1]$ ,  $s_2$  be a separator vertex of  $X_1$  to  $N[a_2]$  such that  $|\{Q \in C(G) : s_i \in Q\}|$  is minimum for  $i = 1, 2$ . By the election of  $y_i$  for  $i = 1, 2$ ,  $s_2 \notin N[a_2]$  and  $s_1 \notin N[a_1]$ . Hence there is a special connection of Type 3 between  $a_1$  and  $a_2$ ; see Figure 18.

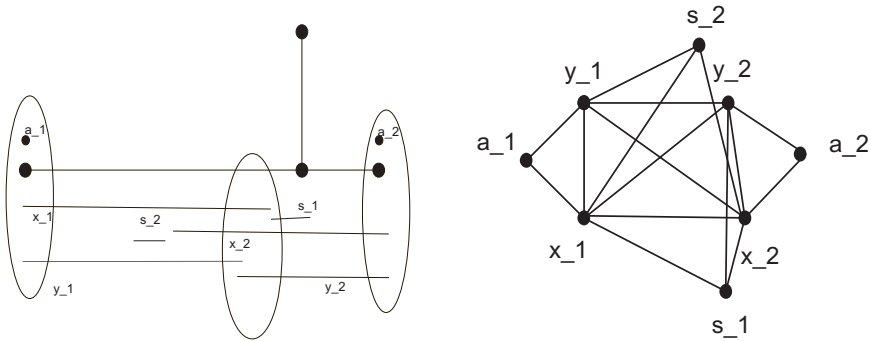


FIGURE 18. Case 3: Type 3 between  $a_1, a_2$ .

We now suppose that  $x_1 \in C$ . Since there is not an edge whose label is contained in  $C$ , then  $y_1 \notin C$ . Hence  $Y_2 \neq C$ . Observe that  $X_2 \neq Y_1$  but  $Y_1$  may be  $Y_2$ . Let  $s_2$  be a separator vertex of  $X_2$  to  $N[a_1]$ ,  $s_1$  be a separator vertex of  $C$  to  $N[a_2]$  such that  $|\{Q \in C(G) : s_i \in Q\}|$  is minimum for  $i = 1, 2$ . As seen on Case 1.2,  $s_2 \notin N[a_1]$  and  $s_1 \notin N[a_2]$ . Hence there is a special connection of Type 3 between  $a_1$  and  $a_2$ ; see Figure 19.

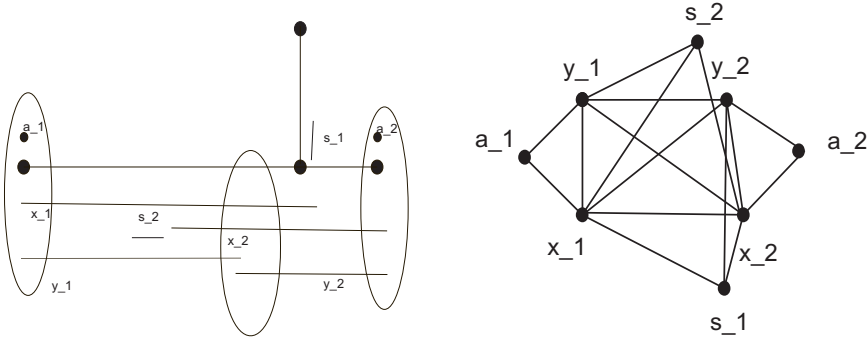


FIGURE 19. Case 3: Type 3 between  $a_1, a_2$ .

In both cases,  $\{a_1, x_1, y_1\}, \{a_2, x_2, y_2\}$  are cliques of  $G$ , and  $\{s_1, s_2, x_1, x_2, y_1, y_2\}$  induces an antenna.

- There is an edge  $e \in T[N[a_1], X_2]$  such that  $lab(e) \subseteq C$ .

**Case 1:**  $T_{x_1} \cap T_{x_2} = \emptyset$ . By the election of  $x_1$ , each vertex in  $lab(e'_1)$  must have  $A_1$  as a leaf. Clearly, there is a path  $P = a_1, y_1, v, y_2, a_2$  in  $G$  between  $a_1$  and  $a_2$ . We will need another path  $Q$  in  $G$  between  $a_1$  and  $a_2$ . Observe that: a) by the election of  $x_1$ , for all  $\bar{e} \in T[X_1, X_2], lab(\bar{e}) \cap N[a_1] = \emptyset$ ; b) by Claim 1 as  $e'_1 \in T[C, N[a_3]]$  is a dominated edge by  $e_1 \in T(N[a_1], C), \forall \bar{e} \in T[X_1, X_2]$ , the vertices in its label cannot have the same ends to  $N[a_1]$ . Hence as  $v \in lab(\bar{e})$  for all  $\bar{e} \in T[X_1, X_2]$ , there is  $w \in lab(\bar{e}) - A_1$ .

As was mentioned above, we need to search vertices for another path. We choose vertices through Election 1 taking  $A = X_1, B = X_2$ , and  $w_i \notin A_1$  for  $i = 1, \dots, n$ . Clearly,  $Q = a_1, x_1, w_1, \dots, w_n, x_2, a_2$  is a path in  $G$  different from  $P$  between  $a_1$  and  $a_2$ . As there is an edge in  $T[X_1, X_2]$  whose label is contained in  $C$  then  $w_n$  is in  $C$ . Clearly  $\{x_2, y_2, w_n, v\}$  is a clique of  $G$ , and  $\{x_1, y_1, v, w_1\}$  may be a clique. Let  $W'Dw_n \in E(T)$  be such that  $Dw_n \in T[C, W']$ . Let  $u$  be a separator vertex of  $W'$  to  $N[a_3]$ , i.e.,  $u \in W' - Dw_n$ . Let  $e(n) = XY$  be the edge which was chosen  $w_n$  with  $Y \in T[X, C]$ . Observe that  $Y$  may be  $X_2$ . On the other hand,  $e(n)$  is dominated by  $Dw_n$  because of the choice of  $w_n$ , which is the shortest vertex to  $C$ , and  $lab(e(n)) \subset C$ . We will analyze if there is another edge  $\tilde{e} \in T[Y, X_2]$  such that  $lab(\tilde{e}) \subset C$ .

□ Suppose that another edge does not exist. By Claim 2, since  $e(n)$  is an edge dominated by  $Dw_n \in T[C, N[a_3]]$ , then for every edge in  $T[W', C]$  its label is not contained in  $Y$ . Hence we choose vertices in label of edges  $\bar{e} \in T[W', C]$  that are not in  $Y$ , through Election 1 taking  $A = W'$  and  $B = C$ . Let  $z_i \notin Y$  be such that  $T_{z_i} = T[Iz_i, Dz_i]$  with  $Iz_i \in T(Y, Dz_i]$  for  $i = 1, \dots, o$ , and  $z_o$  is the last vertex chosen. By the choice of  $z_i \notin Y$  for  $i = 1, \dots, o$ , it follows that  $e'_1 \notin T[W', C]$ .



Now, we will analyze two situations depending on the position of  $Iz_o$  in  $T(Y, C]$ .

First, we consider  $Iz_o \in T(X_2, C]$ . Let  $t$  be a separator vertex of  $X_2$  to  $N[a_1]$  such that  $|\{Q \in C(G) : t \in Q\}|$  is minimum. Observe that  $t \notin X$ . Also  $t$  is not adjacent to  $y_2$  or  $z_o$ . Then, there is a special connection between  $a_1$  and  $a_2$ . More clearly, in case that  $\{x_1, y_1, v, w_1\}$  is not a clique it follows that  $\{a_1, x_1, y_1\}, \{x_1, v, y_1\}, \{x_1, v, w_1\}, \{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}, \{x_2, y_2, a_2\}, \{v, w_n, x_2, y_2, z_o\}, \{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}, \{u, v, z_1\}$  and  $\{v, w_n, x_2, t\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n-2\}$ , and of Type 4 between  $w_{n-1}$  and  $a_2$ ; see Figure 20.

In case that  $\{x_1, y_1, v, w_1\}$  is a clique of  $G$ , by the same argument used in Case 1.2, there are two vertices  $s_1, s_2$  in  $G$  such that  $\{x_1, y_1, v, w_1, s_1, s_2\}$  induces an antenna.

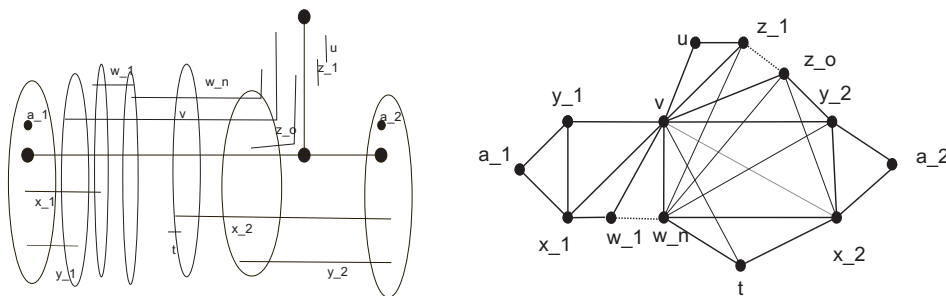


FIGURE 20. Case 1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; Type 4 between  $w_{n-1}$  and  $a_2$ .

Finally, we consider  $Iz_o \in T(Y, X_2]$ . Let  $e(o) = A_o B_o$  be the edge which  $z_o$  was chosen, with  $B_o \in T[C, A_o]$ . Let  $Z'$  be the vertex of  $T$  such that  $Iz_o Z' \in E(T)$  and  $Iz_o \in T[Z', X_2]$ . Observe that  $Z' \neq X$  since  $Iz_o \neq Y$ . Also, by the election of  $z_o$ ,  $e(o)$  is a dominated edge by  $Iz_o$ , then by Claim 2 for every  $e'$  edge in  $T[Z', X_2]$  its label is not contained in  $B_o$ . Hence we choose vertices through Election 1 in label of edges of  $T[Z', X_2]$  such that they are not in  $B_o$ . Let  $t_i \notin B_o$  be the vertices chosen with  $i \in \{1, \dots, p\}$ , and  $t_p$  be the last vertex chosen. It is clear that  $t_p$  may be in  $C$ . But there is not an edge different from  $e(n)$  such that it is contained in  $C$ , then  $t_p \notin C$ . Clearly,  $t_p$  may or may not be adjacent to  $y_2$ .

$\diamond$   $t_p$  is adjacent to  $y_2$ . As  $t_p \notin C$  there is a separator vertex of  $C$  to  $N[a_3]$ . Let  $s_1$  be the separator of  $C$  to  $N[a_3]$  minimizing  $|\{Q \in C(G) : s_1 \in Q\}|$ . As  $z_o$  was chosen instead of  $s_1$  then  $z_{o-1}$  is not adjacent to  $s_1$ . Let  $s$  be a separator of  $X_2$  to  $N[a_1]$  such that  $|\{Q \in C(G) : s \in Q\}|$  is minimum. By the election of  $t_p$ , it is clear that  $s$  is not adjacent to  $t_{p-1}$ . Let  $t$  be a separator of  $It_1$  to  $N[a_1]$  such that  $|\{Q \in C(G) : t \in Q\}|$  is minimum. Observe

that  $t \notin X$ . There is a special connection between  $a_1$  and  $a_2$ . More clearly, in case that  $\{x_1, y_1, w_1, v\}$  is not a clique then  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_1, v, w_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$ ,  $\{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{u, v, z_1\}$ ,  $\{v, w_n, z_o, x_2, t_p, s\}$ ,  $\{t_p, x_2, y_2, z_o, w_n, v\}$ ,  $\{t, w_n, v, t_1\}$ ,  $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1, \dots, p-1}$  and  $\{v, w_n, x_2, y_2, z_o, s_1\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ , and of Type 5 between  $w_{n-1}$  and  $a_2$ ; see Figure 21.

In case that  $\{x_1, y_1, v, w_1\}$  is a clique of  $G$ , as seen on Case 1.2 there are two vertices  $s'_1, s_2$  in  $G$  such that  $\{x_1, y_1, v, w_1, s'_1, s_2\}$  induces an antenna.

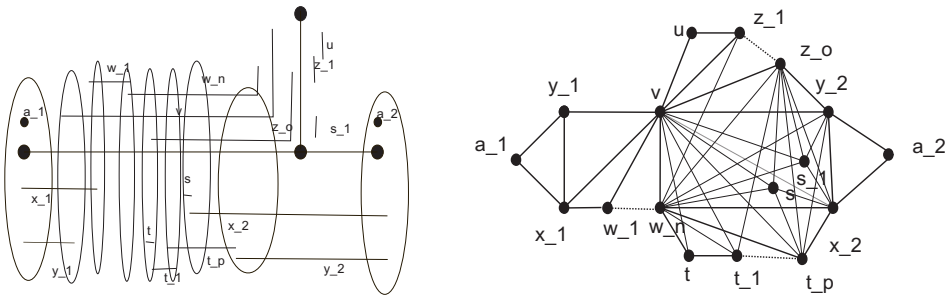


FIGURE 21. Case 1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; of Type 5 between  $w_{n-1}, a_2$ .

$\diamond t_p$  is not adjacent to  $y_2$ . As before, consider only the separator of  $It_1$  to  $N[a_1]$ . Then there is a special connection between  $a_1$  and  $a_2$ . More clearly, in case that  $\{x_1, y_1, w_1, v\}$  is not a clique then  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_1, v, w_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n}$ ,  $\{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{u, v, z_1\}$ ,  $\{t, v, w_n, t_1\}$ ,  $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1, \dots, p-1}$ ,  $\{t_p, x_2, v, w_n, z_o\}$  and  $\{v, w_n, x_2, y_2, z_o\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ , and of Type 6 between  $w_{n-1}$  and  $a_2$ ; see Figure 22.

In case that  $\{x_1, y_1, v, w_1\}$  is a clique of  $G$ , as seen on Case 1.2 there are two vertices  $s_1, s_2$  in  $G$  such that  $\{x_1, y_1, v, w_1, s_1, s_2\}$  induces an antenna.

$\square$  Suppose that there is an edge in  $T[B, X_2]$  which is contained in  $C$ . Let  $\tilde{e}$  be the nearest  $C$ . Observe that  $lab(e(n)) \subset lab(\tilde{e})$ , but by our assumption  $lab(\tilde{e}) \not\subset lab(e(n))$ ; let  $m \in lab(\tilde{e}) - lab(e(n))$  be such that  $T_m$  is the shortest to  $N[a_3]$  with  $Dm$  its leaf in  $T[C, N[a_3]]$ . Clearly,  $m \neq w_n, v$ . Observe that for all edges  $\tilde{e}$  in  $T(\tilde{e}, X_2)$ ,  $lab(\tilde{e}) \not\subset C$ . Therefore there are vertices in the label of edges of  $T(\tilde{e}, X_2)$  that are not in  $C$ .

In case that  $Dm \in T[Dw_n, N[a_3]]$ ,  $\tilde{e}$  is dominated by  $Dw_n$ ; then by Claim 2 there are vertices in the label of edges of  $T[W', C]$  that are not in  $\tilde{B}$  with  $\tilde{e} = \tilde{A}\tilde{B}$  and  $\tilde{B} \in T[\tilde{A}, C]$ . Let  $z_i$  be these vertices for  $i = 1, \dots, o$

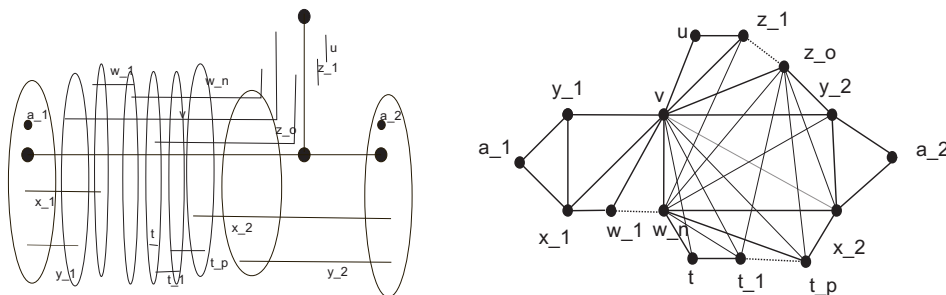


FIGURE 22. Case 1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; Type 6 between  $w_{n-1}, a_2$ .

chosen through Election 1. Therefore  $I_{z_o} \notin \tilde{B}$ . Also, by the election of  $\tilde{e}$ , the vertices  $t_i$  chosen as before are not in  $C$ , in particular the vertex  $t_p \notin C$ . Hence we get situations described previously, i.e., Type 4 or Type 5 or Type 6. More clearly, in case that  $\{x_1, y_1, w_1, v\}$  is not a clique then if  $I_{z_o} \in T(X_2, C)$ ,  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_1, v, w_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{v, w_n, x_2, y_2, z_o\}$ ,  $\{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$ , and  $\{v, w_n, x_2, t\}$  are cliques of  $G$ . If  $I_{z_o} \in T(\tilde{B}, X_2)$  then in case that  $t_p$  is adjacent to  $y_2$ ,  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_1, v, w_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$ ,  $\{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$ ,  $\{v, w_n, z_o, x_2, t_p, s\}$ ,  $\{t_p, x_2, y_2, z_o, w_n, v\}$ ,  $\{t, w_n, v, t_1\}$ ,  $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1, \dots, p-1}$  and  $\{v, w_n, x_2, y_2, z_o, s_1\}$  are cliques of  $G$ . In case that  $t_p$  is not adjacent to  $y_2$  then  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_1, v, w_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n}$ ,  $\{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$ ,  $\{t, v, w_n, t_1\}$ ,  $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1, \dots, p-1}$ ,  $\{t_p, x_2, v, w_n, z_o\}$  and  $\{v, w_n, x_2, y_2, z_o\}$  are cliques of  $G$ . In case that  $\{x_1, y_1, v, w_1\}$  is a clique of  $G$ , as seen on Case 1.2 there are two vertices  $s'_1, s'_2$  in  $G$  such that  $\{x_1, y_1, v, w_1, s'_1, s'_2\}$  induces an antenna.

In case that  $Dm \in T[C, Dw_n]$ ,  $e(n)$  is dominated by  $Dw_n$ ; then by Claim 2 there are vertices chosen through Election 1 that are not in  $Y$ . As before, let  $z_i$  be these vertices for  $i = 1, \dots, o$ . If  $z_o \notin \tilde{B}$  then we get situations described previously. If  $z_o \in \tilde{B}$ , as  $Dm$  dominates  $\tilde{e}$ , no edge of  $T[M', C]$  ( $DmM' \in E(T)$  with  $M' \in T[M, Dw_n]$ ) is dominated by  $\tilde{B}$ , then  $e(o) \notin T[C, M']$ . It is clear that  $I_{z_o} \notin T[X_2, C]$ . Also  $t_p = m$ . In this case  $t_p \in C$ . By Claim 2 as  $\tilde{e}$  is a dominated edge by  $Dt_p = Dm$ , in the label of edges of  $T[C, M']$  there are vertices that are not in  $\tilde{B}$ . Let  $z'_i$  be vertices chosen in the label of edges in  $T[C, M']$  that are not in  $\tilde{B}$  through Election 1 taking  $A = M', B = C$  for  $i = 1, \dots, q$ , and with  $z'_q$  the last vertex chosen. Let  $u'$  be adjacent to  $z'_1$  but not adjacent to  $z'_2$ . If  $z'_q \in T(X_2, C)$  then let  $s$  be a separator of  $X_2$  to  $N[a_1]$  such that  $|\{Q \in C(G) : s \in Q\}|$  is minimum. We obtain a special connection between  $a_1$  and  $a_2$ . More clearly, in case that  $\{x_1, y_1, v, w_1\}$  is not a clique then  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,

$\{x_1, v, w_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$ ,  $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1, \dots, p-1}$ ,  $\{t, v, w_n, t_1\}$ ,  $\{u', z'_1, z_o, w_n, v\}$ ,  $\{s, t_p, w_n, v, z_o, x_2\}$ ,  $\{z'_q, x_2, y_2, v, w_n, t_p, z_o\}$  and  $\{z'_i, z'_{i+1}, v, w_n, t_p, z_o\}_{i=1, \dots, q-1}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ , and of Type 7 between  $w_{n-1}$  and  $a_2$ ; see Figure 23.

In case that  $\{x_1, y_1, v, w_1\}$  is a clique of  $G$ , as seen on Case 1.2 there are two vertices  $s'_1, s'_2$  in  $G$  such that  $\{x_1, y_1, v, w_1, s'_1, s'_2\}$  induces an antenna.

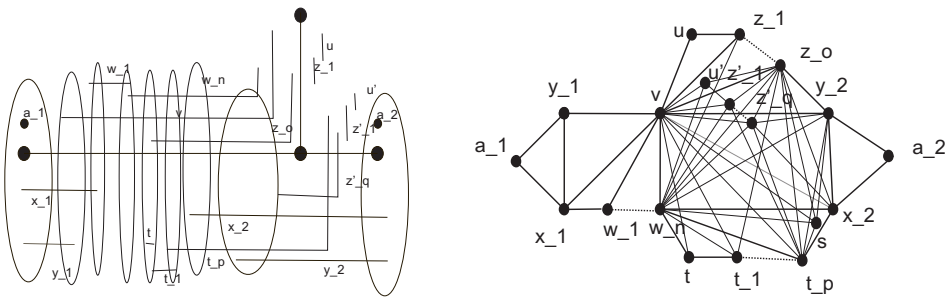


FIGURE 23. Case 1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; Type 7 between  $w_{n-1}, a_2$ .

If  $z'_q \in T(\tilde{B}, X_2]$ , as the edge  $e(q)$  where  $z'_q$  was chosen is dominated by  $Iz'_q$ , it follows by Claim 2 that there are vertices  $t'_i$  that are not in  $B_q$ , with  $e(q) = A_q B_q$  and  $B_q \in T[C, A_q]$ , for  $i = 1, \dots, r$ . Also by the election of  $\tilde{e}$ , they are not in  $C$ . Let  $t'_r$  be the last vertex chosen. Observe that  $t'_r$  may be adjacent to  $y_2$ .

If  $t'_r$  is not adjacent to  $y_2$  then there is a special connection between  $a_1$  and  $a_2$ . More clearly, in case that  $\{x_1, y_1, w_1, v\}$  is not a clique then  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_1, v, w_1\}$ ,  $\{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$ ,  $\{t_i, t_{i+1}, v, w_n, z_o\}_{i=1, \dots, p-1}$ ,  $\{t, v, w_n, t_1\}$ ,  $\{t', t'_1, t_p, v, w_n, z_o\}$ ,  $\{t'_i, t'_{i+1}, z_o, z'_q, t_p, v, w_n\}_{i=1, \dots, r-1}$ ,  $\{t'_r, t_p, v, w_n, z_o, z'_q, x_2\}$ ,  $\{t_p, v, w_n, z_o, z'_q, x_2, y_2\}$ ,  $\{t_p, v, w_n, z_o, z'_i, z'_{i+1}\}_{i=1, \dots, q-1}$  and  $\{u', z'_1, z_o, w_n, v\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n - 2\}$ , and of Type 8 between  $w_{n-1}$  and  $a_2$ ; see Figure 24.

In case that  $\{x_1, y_1, v, w_1\}$  is a clique of  $G$ , as seen on Case 1.2 there are two vertices  $s_1, s_2$  in  $G$  such that  $\{x_1, y_1, v, w_1, s_1, s_2\}$  induces an antenna.

If  $t'_r$  is adjacent to  $y_2$ , let  $s$  be a separator of  $X_2$  to  $N[a_1]$  such that  $|\{Q \in C(G) : s \in Q\}|$  is minimum. By the election of  $t'_r$ , it is clear that  $s$  is not adjacent to  $t'_{r-1}$ ; recall that  $t'_r \notin C$ . Then, let  $s_1$  be a separator vertex of  $C$  to  $N[a_3]$  such that  $|\{Q \in C(G) : s_1 \in Q\}|$  is minimum.

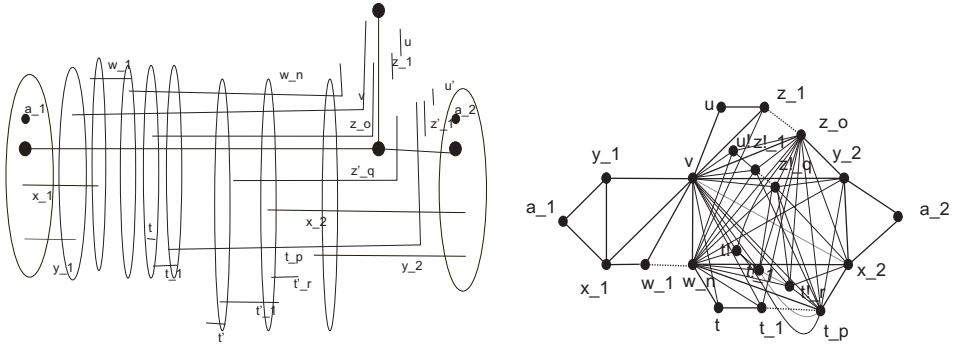


FIGURE 24. Case 1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; Type 8 between  $w_{n-1}, a_2$ .

Observe that  $s_1 \neq z'_q$  since  $z'_q \in T(\tilde{B}, X_2]$ . Hence there is a special connection between  $a_1$  and  $a_2$ . More clearly, in case that  $\{x_1, y_1, w_1, v\}$  is not a clique then  $\{a_1, x_1, y_1\}, \{x_1, v, y_1\}, \{x_1, v, w_1\}, \{w_i, v, w_{i+1}\}_{i=1, \dots, n-1}, \{x_2, y_2, a_2\}, \{z_i, z_{i+1}, v, w_n\}_{i=1, \dots, o-1}, \{u, v, z_1\}, \{t_i, t_{i+1}, v, w_n, z_o\}_{i=1, \dots, p-1}, \{t, v, w_n, t_1\}, \{t', t'_1, t_p, v, w_n, z_o\}, \{t'_i, t'_{i+1}, z_o, z'_q, t_p, v, w_n\}_{i=1, \dots, r-1}, \{t'_r, t_p, v, w_n, z_o, z'_q, x_2, y_2, s_1\}, \{t_p, v, w_n, z_o, z'_i, z'_{i+1}\}_{i=1, \dots, q-1}, \{t_p, v, w_n, z_o, z'_1, u'\}$  and  $\{s, t'_r, z'_q, t_p, z_o, v, w_n, x_2\}$  are cliques of  $G$ .

Observe that there is a special connection of Type 2 between  $a_1, w_2$ , of Type 1 between  $w_i, w_{i+2}$  with  $i \in \{1, \dots, n-2\}$ , and of Type 9 between  $w_{n-1}$  and  $a_2$ ; see Figure 25.

In case that  $\{x_1, y_1, v, w_1\}$  is a clique of  $G$ , as seen on Case 1.2 there are two vertices  $s'_1, s_2$  in  $G$  such that  $\{x_1, y_1, v, w_1, s'_1, s_2\}$  induces an antenna.

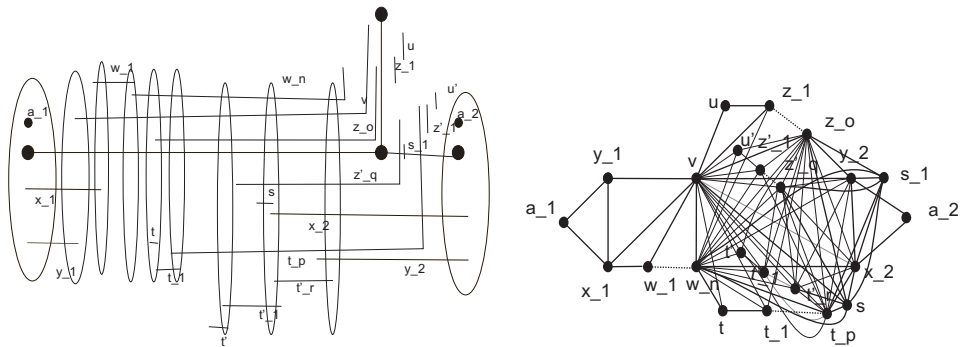


FIGURE 25. Case 1: Type 2 between  $a_1, w_2$ ; Type 1 between  $w_i, w_{i+2}$ ; Type 9 between  $w_{n-1}, a_2$ .

**Case 2:**  $T_{x_1} \cap T_{x_2} \cap T_{y_1} \neq \emptyset$  but  $T_{y_2} \cap T_{x_1} = \emptyset$ , or  $T_{x_1} \cap T_{x_2} \cap T_{y_2} \neq \emptyset$  but  $T_{y_1} \cap T_{x_2} = \emptyset$ . By our assumption, there is an edge in  $T[N[a_1], X_2]$  whose label is contained in  $C$ . Hence  $T_{x_1} \cap T_{x_2} \cap T_{y_1} = \emptyset$ .

Clearly, there are two paths in  $G$  between  $a_1$  and  $a_2$ ;  $P = a_1, x_1, x_2, a_2$  and  $Q = a_1, y_1, v, y_2, a_2$ . On the other hand,  $\{x_1, x_2, y_2, v\}$  is a clique of  $G$ . In this situation,  $x_1$  may be in  $lab(e'_1)$ . We know that there is an edge in  $T[Y_1, X_2]$  whose label is contained in  $C$ . Let  $\tilde{e} = \tilde{A}\tilde{B}$  be the nearest  $C$  with  $\tilde{B} \in T[\tilde{A}, C]$ , and  $m \in lab(\tilde{e})$  such that  $T_m$  is the shortest to  $N[a_3]$  and  $Dm$  its leaf in  $T[C, N[a_3]]$ . Observe that  $m$  is not  $v$ . Moreover,  $Dm \neq B'_1$ ; otherwise  $T' = T - \{e'_1, \tilde{e}\} + \tilde{A}B'_1 + A'_1\tilde{B}$  is a DV-model that can be rooted on  $N[a_3]$ , a contradiction.

On the other hand, we choose  $w_i$  in label of edges in  $T[Y_1, X_2]$  with the Election 1 taking  $A = Y_1$  and  $B = X_2$ . Let  $w_n$  be the last vertex chosen, and  $Dw_n$  be the leaf of  $w_n$  to  $N[a_3]$ . Observe that  $w_n$  may be  $x_1$  or  $m$ .

If  $m = x_1$ , let  $X'_1 X_1$  be the edge of  $T$  with  $X'_1 \in T[X_1, N[a_3]]$ . As  $X_1$  dominates  $\tilde{e}$  it follows by Claim 2 that for all edges  $\bar{e} \in T[C, X'_1]$ ,  $lab(\bar{e}) \not\subseteq \tilde{B}$ . Then we choose vertices in the label of edges in  $T[C, X'_1]$  through Election 1 with  $A = X'_1$  and  $B = C$  such that they are not in  $\tilde{B}$ . Let  $z_i$  be these vertices chosen for  $i = 1, \dots, o$ , and  $z_o$  be the last vertex chosen. As in Case 1, we will analyze whether  $Iz_o$  is in  $T(\tilde{B}, X_2]$ , and we obtain a special connection of Type 4, 5, or 6, taking  $x_1$  instead of  $w_i$  in Case 1. More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{v, x_1, x_2, y_2, z_o\}$ ,  $\{z_i, z_{i+1}, v, x_1\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$  and  $\{v, x_1, x_2, t\}$  are cliques of  $G$ ; so there is a special connection of Type 4 between  $y_1, a_2$ , and of Type 1 between  $a_1, v$ ; see Figure 26.

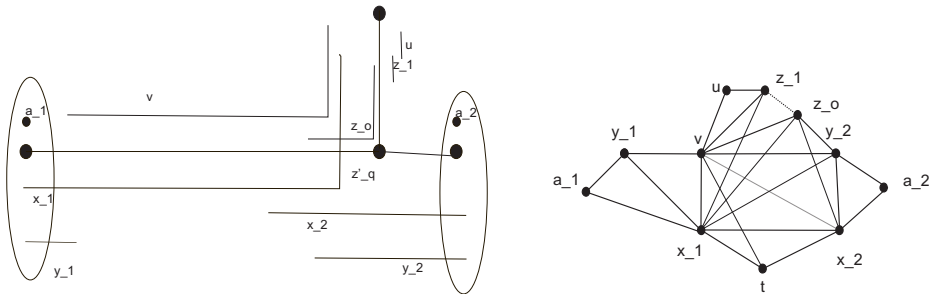


FIGURE 26. Case 2: Type 4 between  $y_1, a_2$  and Type 1 between  $a_1, v$ .

Or  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{z_i, z_{i+1}, v, x_1\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$ ,  $\{v, x_1, z_o, x_2, t_p, s\}$ ,  $\{t_p, x_2, y_2, z_o, x_1, v\}$ ,  $\{t, x_1, v, t_1\}$ ,  $\{t_i, t_{i+1}, v, x_1, z_o\}_{i=1, \dots, p-1}$  and  $\{v, x_1, x_2, y_2, z_o, s_1\}$  are cliques of  $G$ ; so there is a special connection of Type 5 between  $y_1, a_2$  and of Type 1 between  $a_1, v$ . Or  $\{a_1, x_1, y_1\}$ ,  $\{x_1, v, y_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{z_i, z_{i+1}, v, x_1\}_{i=1, \dots, o-1}$ ,  $\{u, v, z_1\}$ ,  $\{t, v, x_1, t_1\}$ ,  $\{t_i, t_{i+1}, v, x_1, z_o\}_{i=1, \dots, p-1}$ ,  $\{t_p, x_2, v, x_1, z_o\}$  and  $\{v, x_1, x_2,$

$y_2, z_o\}$  are cliques of  $G$ ; so there is a special connection of Type 6 between  $y_1, a_2$ , and of Type 1 between  $a_1, v$ ; see Figure 27.

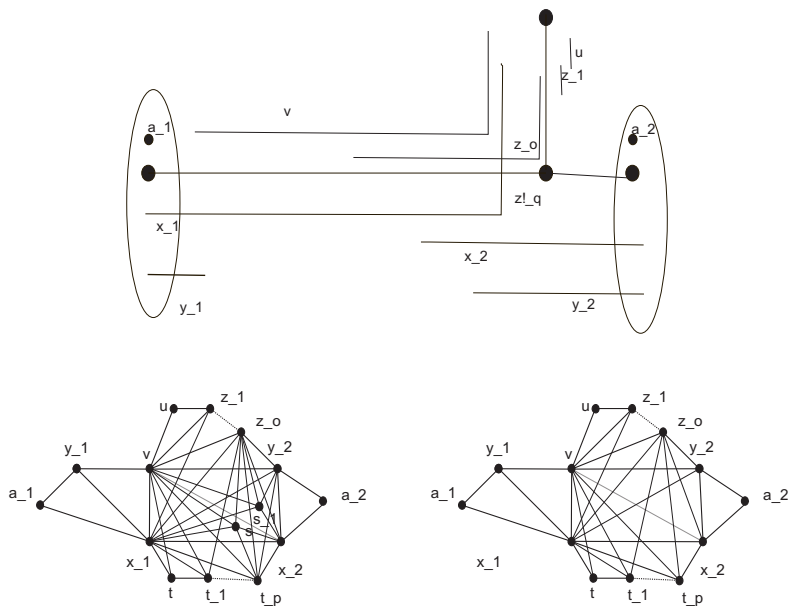


FIGURE 27. Case 2: Type 5 between  $y_1, a_2$  and Type 1 between  $a_1, v$  or Type 6 between  $y_1, a_2$  and Type 1 between  $a_1, v$ .

If  $m = w_n$ , let  $P' = a_1, y_1, w_1, \dots, w_n, x_2, a_2$  and  $Q' = a_1, x_1, y_2, a_2$  be paths in  $G$  between  $a_1$  and  $a_2$ . Hence there is a special connection of Type 4 or Type 5 or Type 6 between  $w_{n-1}$  and  $a_2$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{w_1, x_1, y_1\}$ ,  $\{w_i, w_{i+1}, x_1\}_{i=1, \dots, n-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{t, x_2, w_n, x_1\}$ ,  $\{z_o, x_1, w_n, x_2, y_2\}$ ,  $\{z_i, z_{i+1}, w_n, x_1\}_{i=1, \dots, o-1}$  and  $\{z_1, x_1, u\}$  are cliques of  $G$ . Or  $\{a_1, x_1, y_1\}$ ,  $\{w_1, x_1, y_1\}$ ,  $\{w_i, w_{i+1}, x_1\}_{i=1, \dots, n-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{t_p, y_2, x_2, z_o, w_n, x_1\}$ ,  $\{x_2, t_p, z_o, s, w_n, x_1\}$ ,  $\{z_o, t_i, t_{i+1}, x_1, w_n\}_{i=1, \dots, p-1}$ ,  $\{t, w_n, x_1, t_1\}$ ,  $\{z_o, x_1, w_n, x_2, y_2\}$ ,  $\{z_i, z_{i+1}, w_n, x_1\}_{i=1, \dots, o-1}$  and  $\{z_1, x_1, u\}$  are cliques of  $G$ . Or  $\{a_1, x_1, y_1\}$ ,  $\{w_1, x_1, y_1\}$ ,  $\{w_i, w_{i+1}, x_1\}_{i=1, \dots, n-1}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{y_2, x_2, z_o, w_n, x_1\}$ ,  $\{x_2, t_p, z_o, w_n, x_1\}$ ,  $\{z_o, t_i, t_{i+1}, x_1, w_n\}_{i=1, \dots, p-1}$ ,  $\{t, w_n, x_1, t_1\}$ ,  $\{z_i, z_{i+1}, w_n, x_1\}_{i=1, \dots, o-1}$  and  $\{z_1, x_1, u\}$  are cliques of  $G$ ; see Figure 28.

If  $m \neq x_1, w_n$ , let  $P' = a_1, y_1, w_1, \dots, w_n, x_2, a_2$  and  $Q' = a_1, x_1, y_2, a_2$  be paths in  $G$  between  $a_1$  and  $a_2$ . Hence there is a special connection of Type 4 or Type 5 or Type 6 or Type 7 or Type 8 or Type 9 between  $w_{n-1}$  and  $a_2$  considering the analysis of Case 1 when there is another edge contained in  $C$ .

**Case 3:**  $T_{x_1} \cap T_{x_2} \cap T_{y_1} \cap T_{y_2} \neq \emptyset$ . Clearly, there are two paths in  $G$  between  $a_1$  and  $a_2$ . Let  $P = a_1, y_1, y_2, a_2$  and  $Q = a_1, x_1, x_2, a_2$  be paths

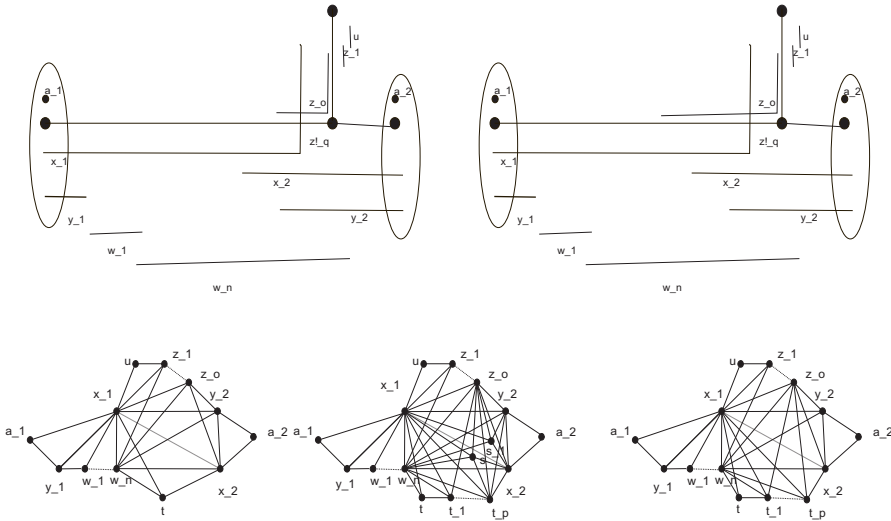


FIGURE 28. Case 2: Type 4 or Type 5 or Type 6 between  $w_{n-1}$ ,  $a_2$  and Type 1 between  $a_1, w_1$ .

in  $G$ . Also  $\{x_1, y_1, x_2, y_2\}$  is a clique of  $G$ . By the existence of an edge  $e \in T[N[a_1], X_2]$  such that  $lab(e) \subset C$ , we have  $x_1 \in C$  and  $y_1 \in C$ . Observe that  $N[a_1]A_1$  has a label contained in  $C$ , and may have another edge. On the other hand,  $x_1$  and  $y_1$  cannot be both vertices of  $lab(e'_1)$ , otherwise  $T' = T - \{N[a_1]A_1, e'_1\} + N[a_1]B'_1 + A'_1A_1$  is a DV-model of  $G$  rooted on  $N[a_3]$ , a contradiction. Hence  $y_1 \notin lab(e'_1)$ ; moreover,  $y_1 \notin B'_1$ . Let  $Y'_1Y_1 \in E(T)$  be such that  $Y'_1 \in T[Y_1, N[a_3]]$ , and let  $\tilde{e} = \tilde{A}\tilde{B}$  be the closest edge to  $C$  dominated by  $Y_1$ . Clearly  $lab(\tilde{e}) \subset C$ . By Claim 2, for all  $e' \in T[C, Y'_1]$  we have  $lab(e') \not\subseteq \tilde{B}$ . Clearly, there are vertices  $z_i \in lab(e') - \tilde{B}$  which were chosen through Election 1, and if  $z_o$  is the last vertex chosen then analyzing where  $Iz_o$  is we obtain the situations described in Case 1, i.e., there is a special connection of Type 4 or Type 5 or Type 6 taking  $w_n = y_1$  and  $v = x_1$ . More clearly,  $\{a_1, x_1, y_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{t, x_2, y_1, x_1\}$ ,  $\{z_o, x_1, y_1, x_2, y_2\}$ ,  $\{z_i, z_{i+1}, y_1, x_1\}_{i=1, \dots, o-1}$  and  $\{z_1, x_1, u\}$  are cliques of  $G$ . Or  $\{a_1, x_1, y_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{t_p, y_2, x_2, z_o, y_1, x_1\}$ ,  $\{x_2, t_p, z_o, s, y_1, x_1\}$ ,  $\{z_o, t_i, t_{i+1}, x_1, y_1\}_{i=1, \dots, p-1}$ ,  $\{t, y_1, x_1, t_1\}$ ,  $\{z_o, x_1, y_1, x_2, y_2\}$ ,  $\{z_i, z_{i+1}, y_1, x_1\}_{i=1, \dots, o-1}$  and  $\{z_1, x_1, u\}$  are cliques of  $G$ . Or  $\{a_1, x_1, y_1\}$ ,  $\{x_2, y_2, a_2\}$ ,  $\{y_2, x_2, z_o, y_1, x_1\}$ ,  $\{x_2, t_p, z_o, y_1, x_1\}$ ,  $\{z_o, t_i, t_{i+1}, x_1, y_1\}_{i=1, \dots, p-1}$ ,  $\{t, y_1, x_1, t_1\}$ ,  $\{z_i, z_{i+1}, y_1, x_1\}_{i=1, \dots, o-1}$  and  $\{z_1, x_1, u\}$  are cliques of  $G$ ; see Figure 29.  $\square$

The following corollary allows us to construct forbidden induced subgraphs for rooted directed path graphs different from those described in [1].



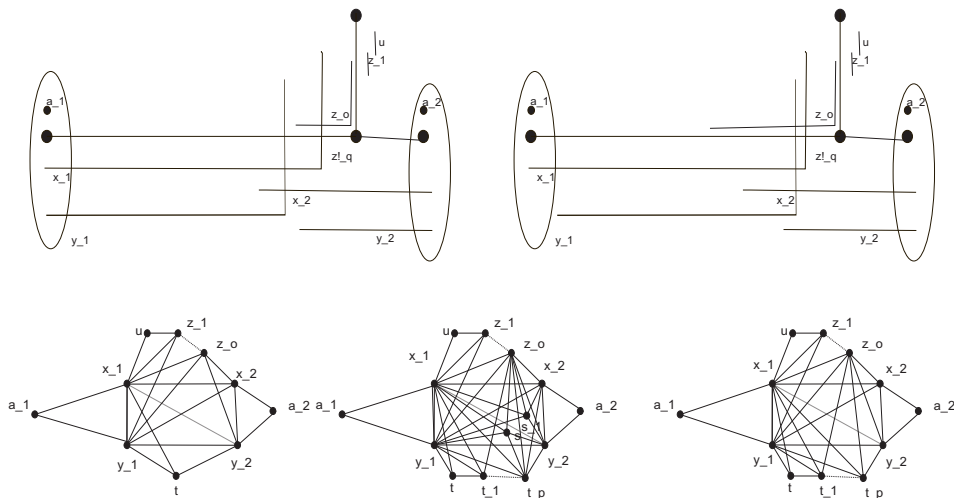


FIGURE 29. Case 3: Type 4 or Type 5 or Type 6 between  $a_1, a_2$ .

**Corollary 2.** *Let  $G$  be a DV graph with an asteroidal quadruple  $\{a_1, a_2, a_3, a_4\}$ . If  $a_1, a_2$  and  $a_3, a_4$  are linked by a special connection then  $G$  is not an RDV graph.*

*Proof.* Let  $Q_{a_i}$  be a clique that contains  $a_i$  for  $i = 1, 2, 3, 4$  and  $T$  be a DV-model of  $G$ . Since  $a_1, a_2, a_3, a_4$  is an asteroidal quadruple,  $T[Q_{a_1}, Q_{a_2}, Q_{a_3}, Q_{a_4}]$  has four leaves. By Theorem 1,  $T(a_1, a_2)$  and  $T(a_3, a_4)$  are directed path, then  $T$  cannot be rooted. Therefore  $G$  is not an RDV graph.  $\square$

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