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A DISCUSSION ON THE NATURAL DOMAIN OF RADIAL TOEPLITZ OPERATORS IN SEGAL-BARGMANN SPACE

Romina A. Ramírez[1 §], Gerardo L. Rossini[2], Marcela Sanmartino[3]
[1,3]Departamento de Matemática
Facultad de Ciencias Exactas
Universidad Nacional de La Plata
ARGENTINA
[2]IFLP-CONICET and Departamento de Física
Facultad de Ciencias Exactas
Universidad Nacional de La Plata
ARGENTINA

Abstract: We discuss the domain of Toeplitz operators with radial symbols in the Segal-Bargmann space: we point out and correct missleading staments in previous works, establishing the conditions under which a given Toeplitz operator is unitarily equivalent to a diagonal operator in the space $l^2(\mathbb{C})$ of square summable complex sequences.

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1. Introduction

Toeplitz operators were introduced in physics by Berezin [1] [2] in the context

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§Correspondence author

of quantization procedures, *i.e.* in the association of a classical function in phase space with a quantum observable. In this context Toeplitz operators on the Segal-Bargmann space $F^2(\mathbb{C}, d\mu)$ arise as a particular example of a more general concept, that of linear operators with anti-Wick or contravariant symbols defined on a Fock space [3].

Given a measurable function $\varphi(z, \overline{z})$, the action of the Toeplitz operator T_φ with symbol $\varphi(z, \overline{z})$ on a function $f(z) \in F^2(\mathbb{C}, d\mu)$ is defined as the Bargmann projection of the pointwise product φf. The operator has a natural domain restricted to functions f such that the Bargmann projection of φf is well defined. For the present discussion, notice that such domain may not be dense in the Segal-Bargmann space, leading to a not well defined linear operator. This fact poses a problem on various properties of Toeplitz operators, such as composition closedness.

Well defined Toeplitz operators with radial symbols $\varphi(|z|)$ are specially simple, and have been used elsewhere for elementary examples [4]. The underlying reason is that such operators, under the unitary isometry relating the Segal-Bargmann space with the space $l^2(\mathbb{C})$ of complex square integrable sequences, are mapped into diagonal operators in the canonical orthonormal basis. Recently, Grudsky and Vasilevski [5] presented a detailed description of this mapping and the corresponding behaviour of radial Toeplitz operators with the aim of relating operator properties such as boundness and compactness to spectral properties. However, not any radial symbol gives rise to a well defined Toeplitz operator. We find that the class of symbols considered by these authors does not guarantee the equivalence between radial Toeplitz operators and diagonal operators on $l^2(\mathbb{C})$. This simple fact has been the source of erroneous assertions in [5] and subsequent papers (see for instance [6]). A striking situation where the claimed equivalence fails can be constructed in the form of a Toeplitz operator with trivial natural domain (only the null vector), related to a diagonal operator with dense domain in $l^2(\mathbb{C})$.

In the present work we review the Toeplitz operator facts mentioned above and establish necessary and sufficient conditions for the unitary equivalence of a radial Toeplitz operator on the Segal-Bargmann space with a diagonal operator on $l^2(\mathbb{C})$. We employ the Fock space formalism, usual in modern quantum mechanics, in order to present a unified description of unitarily equivalent operators on $F^2(\mathbb{C}, d\mu)$ and $l^2(\mathbb{C})$. The results are illustrated with simple examples, to be contrasted with statements in Ref. [5].

2. Preliminaries: Toeplitz Operators and Anti-Wick Operators

2.1. Toeplitz Operators

We denote $L^2(\mathbb{C}, d\mu)$ the Hilbert space of complex functions $g(z, \overline{z})$ on the complex plane $\mathbb{C}$, not necessarily analytical, which are square integrable with respect to the Gaussian measure $d\mu(z) = \frac{1}{\pi}e^{-|z|^2}dz\, d\overline{z}$ (being $dz\, d\overline{z}$ the Lebesgue measure on $\mathbb{C}$). The Segal-Bargmann space, denoted here $F^2(\mathbb{C}, d\mu)$, is the Hilbert subspace of $L^2(\mathbb{C}, d\mu)$ containing the square integrable *analytic* functions $f(z)$ on $\mathbb{C}$, with respect to the same measure.

The Bargmann projection P relates $L^2(\mathbb{C}, d\mu)$ and $F^2(\mathbb{C}, d\mu)$. It is defined on functions $g \in L^2(\mathbb{C}, d\mu)$, with the help of the Poisson vectors

$$K_{\overline{z}}(w) := e^{\overline{z}w}, \tag{2.1}$$

as the inner product

$$P(g)(z) = (g, K_{\overline{z}}) \equiv \int_{\mathbb{C}} e^{z\overline{w}} g(z) d\mu(z). \tag{2.2}$$

The Poisson vectors form an overcomplete set in $F^2(\mathbb{C}, d\mu)$, then the Bargmann projection maps $L^2(\mathbb{C}, d\mu)$ into $F^2(\mathbb{C}, d\mu)$, and acts as the identity operator in $F^2(\mathbb{C}, d\mu)$. These features are essential for the definition of Toeplitz operators (see for instance [7]):

Definition 1. Given a measurable function $\varphi(z, \overline{z})$ defined on $\mathbb{C}$, not necessarily analytic, a Toeplitz operator with symbol $\varphi(z, \overline{z})$ is a linear operator T_φ in the Segal-Bargmann space, acting on $f(z) \in F^2(\mathbb{C}, d\mu)$ as

$$T_\varphi(f)(z) = P(\varphi f)(z), \tag{2.3}$$

where P is the Bargmann projection.

In other words, the action of the Toeplitz operator on a function in the Segal-Bargmann space is obtained as the Bargmann projection of the pointwise multiplication of its symbol with the function. An explicit integral expression reads

$$T_\varphi(f)(z) = \int_{\mathbb{C}} d\mu(w) e^{z\overline{w}} \varphi(w, \overline{w}) f(w). \tag{2.4}$$

For the aim of the present note it is crucial that, given a symbol φ, the Bargmann projection in (2.3) may be not well defined for any $f \in F^2(\mathbb{C}, d\mu)$: the sufficient and necessary condition on f is that $\varphi f \in L^2(\mathbb{C}, d\mu)$. We have the following (see for instance [1]):

Definition 2. Given a measurable function $\varphi(z,\overline{z})$ defined on $\mathbb{C}$, the *natural domain* for the Toeplitz operator with symbol φ is the subset of $F^2(\mathbb{C},d\mu)$

$$Dom(T_\varphi) = \{f \in F^2(\mathbb{C},d\mu) : \varphi f \in L^2(\mathbb{C},d\mu)\}. \tag{2.5}$$

Remark. An endomorphism in a Hilbert space is said to be well defined if its domain is dense in this space. Accordingly, we will say that the Toeplitz operator T_φ is *well defined* only when its natural domain is dense in $F^2(\mathbb{C},d\mu)$.

In order to deal with well defined Toeplitz operators, one must precise the classes of symbols under consideration. We recall for instance that Berger and Coburn [8] have developed a symbolic calculus for Toeplitz operators with bounded symbols, whose natural domains are the whole space $F^2(\mathbb{C},d\mu)$. More generally, Folland [9] considers a class containing unbounded symbols φ characterized by

$$\exists\, C > 0, \delta < 1/2, \text{ such that } \quad |\varphi(z,\overline{z})| \le C \exp(\delta|z|^2). \tag{2.6}$$

Later, Coburn [4] defined a still more general class of symbols φ satisfying

$$\forall z \in \mathbb{C}, \quad \varphi K_{\overline{z}} \in L^2(\mathbb{C},d\mu)\,, \tag{2.7}$$

that is requiring that all the Poisson vectors belong to the natural domain of T_φ.

In order to establish a natural unitary transformation between Toeplitz operators and linear operators acting on complex sequences, we will resort in the following to the definition of anti-Wick operators acting on an abstract Fock space. This framework will also help in making apparent whether a Toeplitz operator is well defined.

2.2. Fock Space Structure of $F^2(\mathbb{C},d\mu)$

We use Dirac's notation, usual in quantum physics, to distinguish abstract Hilbert space elements from their concrete realizations as functions or sequences. We also take the opportunity to present a brief summary of this notation and its jargon (see Table 1).

A Fock space $\mathcal{F}$ (with one *degree of freedom*) is a Hilbert space containing a unitary vector $|0\rangle$ (called *vacuum vector* in quantum physics), endowed with an operator $\widehat{a}$ such that $\widehat{a}|0\rangle = 0$ (called *annihilation* operator), and its adjoint $\widehat{a}^\dagger$ (called *creation* operator), satisfying the canonical commutation rules $\widehat{a}\widehat{a}^\dagger -$

	usual notation	Dirac's notation	Dirac's glossary
vector	ψ	$\|\psi\rangle$	*"ket"*
dual linear form	ψ^*	$\langle\psi\|$	*"bra"*
inner product (linear in ψ)	(ψ, η)	$\langle\eta\|\psi\rangle$	*"bracket"*
orthogonal projector	P_ψ	$\|\psi\rangle\langle\psi\|$	*"ket-bra"*
linear operator	$T\psi$	$T\|\psi\rangle$	
	$(T\psi, \eta)$	$\langle\eta\|T\|\psi\rangle$	

Table 1: Dirac's notation for Hilbert spaces

$\widehat{a}^\dagger\widehat{a} = \mathbb{I}$, with $\mathbb{I}$ the identity operator in $\mathcal{F}$[1]. From these elements one can construct the infinite numerable set of vectors

$$\left\{ |n\rangle = \frac{(\widehat{a}^\dagger)^n |0\rangle}{\sqrt{n!}} \right\}_{n\in\mathbb{N}}, \tag{2.8}$$

which is orthonormal and is assumed to be complete in the Hilbert space. Completeness provides a *resolution of the identity*

$$\mathbb{I} = \sum_{n\in\mathbb{N}} |n\rangle\langle n| \tag{2.9}$$

where the sum is understood in weak sense, meaning that for any pair of vectors $|\psi\rangle$, $|\eta\rangle$ in the Fock space, $\langle\eta|\psi\rangle = \sum_{n\in\mathbb{N}}\langle\eta|n\rangle\langle n|\psi\rangle$.

The space $F^2(\mathbb{C}, d\mu)$ has the structure of a Fock space with one degree of freedom. The vacuum vector can be chosen as the constant function $\phi_0(z) = 1$, while the annihilation and creation operators are given by the linear operators

$$\begin{aligned} \widehat{a} &= \frac{\partial}{\partial z}, \\ \widehat{a}^\dagger &= z. \end{aligned} \tag{2.10}$$

The recursive action of the creation operator $\widehat{a}^\dagger$ on the vacuum vector generates an infinite numerable set in $F^2(\mathbb{C}, d\mu)$

$$\left\{ \frac{(\widehat{a}^\dagger)^n \phi_0}{\sqrt{n!}} \right\}_{n\in\mathbb{N}} = \left\{ \frac{z^n}{\sqrt{n!}} \right\}_{n\in\mathbb{N}}. \tag{2.11}$$

[1]This Fock space provides a representation space for the Heisenberg-Weyl Lie algebra, generated by $\left\{\mathbb{I}, \widehat{a}, \widehat{a}^\dagger\right\}$.

This set is complete, as it provides the expansion of any square integrable analytic function as a Taylor expansion.

Remark. We will refer by *abstract* Fock space to the Hilbert space defined from a vacuum vector and a pair of annihilation and creation operators (the latter adjoint to the former with respect to some inner product) satisfying the canonical commutation relations. Dirac's notation in Table 1 will be used for vectors, inner products and operators in the abstract Fock space. A concrete example of a Hilbert space with such structure, such as $F^2(\mathbb{C}, d\mu)$, will be called in the following a *realization* of the abstract Fock space. In this sense, the function $\phi_0(z)$ realizes the vacuum vector $|0\rangle$, $\frac{\partial}{\partial z}$ realizes the operator $\widehat{a}$, etc., in $F^2(\mathbb{C}, d\mu)$.

Amongst the abstract Fock space $\mathcal{F}$ and any of its realizations there exists a natural unitary isomorphism, that described by the identification of the vacuum vectors and the action of respective creation and annihilation operators. Regarding the realization in $F^2(\mathbb{C}, d\mu)$, we will denote by $U : \mathcal{F} \to F^2(\mathbb{C}, d\mu)$ such isomorphism and call $\psi(z) = U(|\psi\rangle)$ the realization of $|\psi\rangle$. An explicit expression can be given for $\psi(z)$ in terms of Taylor expansions, but we will provide a more convenient form in the next subsection.

2.3. Fock Space Coherent States

The realization of a Fock space vector $|\psi\rangle$ as a function $\psi(z) \in F^2(\mathbb{C}, d\mu)$ can be obtained by means of a construction closely related to the Bargmann projection. In order to present this construction we introduce the so called coherent states (see for instance [10]).

In a Fock space $\mathcal{F}$ with one degree of freedom one can define a set of vectors, called *coherent states*, by

$$\left\{ |z\rangle = e^{-|z|^2/2} e^{z\widehat{a}^\dagger} |0\rangle \right\}_{z \in \mathbb{C}}, \tag{2.12}$$

where $e^{z\widehat{a}^\dagger}$ stands for the series expansion $\sum_n \frac{1}{n!} z^n (\widehat{a}^\dagger)^n$. One can readily prove that the coherent states $|z\rangle$ are the normalized eigenvectors of the operator $\widehat{a}$ labeled by their eigenvalues, $\widehat{a}|z\rangle = z|z\rangle$.

The set of coherent states is not an orthogonal basis but forms an overcomplete set in $\mathcal{F}$. The completeness is given by the *resolution of the identity* in $\mathcal{F}$

$$\mathbb{I} = \int_{\mathbb{C}} \frac{dz\, d\overline{z}}{\pi} |\overline{z}\rangle\langle\overline{z}|, \tag{2.13}$$

with the integral understood in the weak sense, *i.e.*

$$\langle\eta|\psi\rangle = \int_{\mathbb{C}} \frac{dz\,d\overline{z}}{\pi}\langle\eta|\overline{z}\rangle\langle\overline{z}|\psi\rangle \tag{2.14}$$

for any pair of vectors $|\psi\rangle$, $|\eta\rangle$ in the Fock space. From the the canonical commutation rules one can show that coherent states are not orthogonal but

$$\langle z|w\rangle = e^{-\frac{1}{2}|z|^2}e^{-\frac{1}{2}|w|^2}e^{\overline{z}w}, \tag{2.15}$$

implying that the set (2.12) is overcomplete. A general expansion of a vector $|\psi\rangle$ in terms of coherent states is conveniently written as

$$|\psi\rangle = \int_{\mathbb{C}} \frac{dz\,d\overline{z}}{\pi}e^{-\frac{1}{2}|z|^2}f(z,\overline{z})\,|\overline{z}\rangle \tag{2.16}$$

where, as before, it must be understood that

$$\langle\eta|\psi\rangle = \int_{\mathbb{C}} \frac{dz\,d\overline{z}}{\pi}e^{-\frac{1}{2}|z|^2}f(z,\overline{z})\langle\eta|\overline{z}\rangle \tag{2.17}$$

for any vector $|\eta\rangle$ in $\mathcal{F}$. It is important to notice that $\langle\psi|\psi\rangle < \infty$ if and only if $f(z,\overline{z}) \in L^2(\mathbb{C}, d\mu)$.

Overcompletness implies that, for a given vector $|\psi\rangle$, the function $f(z,\overline{z})$ in (2.16) is not unique. We can then consider an equivalence relation in $L^2(\mathbb{C}, d\mu)$: two functions are equivalent if they expand the same vector in (2.16). In order to characterize the equivalence class of functions in $L^2(\mathbb{C}, d\mu)$ expanding a a given vector $|\psi\rangle$, we compute from (2.16) the projections $\langle\overline{z}|\psi\rangle$ on coherent states obtaining

$$\langle\overline{z}|\psi\rangle = e^{-\frac{1}{2}|z|^2}P(f)(z), \tag{2.18}$$

where $P(f)$ is the analytic Bargmann projection (2.2) of f. We then use (2.13) to recover

$$|\psi\rangle = \int_{\mathbb{C}} \frac{dz\,d\overline{z}}{\pi}|\overline{z}\rangle\langle\overline{z}|\psi\rangle = \int_{\mathbb{C}} \frac{dz\,d\overline{z}}{\pi}e^{-\frac{1}{2}|z|^2}P(f)(z)|\overline{z}\rangle, \tag{2.19}$$

showing that for a given vector $|\psi\rangle$ there exists an analytic square integrable function $\psi(z) := P(f)(z)$ expanding it in the coherent states set. Moreover, as $P^2 = P$, this function $\psi(z)$ is unique. With the help of (2.9) one can build the power series expansion

$$\psi(z) = \sum_{n\in\mathbb{N}}\langle n|\psi\rangle\frac{z^n}{\sqrt{n!}} \tag{2.20}$$

showing that $\psi(z)$ is the realization of $|\psi\rangle$ mentioned in the previous subsection. We can then define the linear transformation $U : \mathcal{F} \to F^2(\mathbb{C}, d\mu)$ by,

$$\psi(z) = U(|\psi\rangle) = e^{\frac{1}{2}|z|^2} \langle \overline{z}|\psi\rangle \tag{2.21}$$

It is invertible, with inverse read from (2.19) as

$$|\psi\rangle = U^{-1}(\psi(z)) = \int_{\mathbb{C}} \frac{dz\, d\overline{z}}{\pi} e^{-\frac{1}{2}|z|^2} \psi(z) |\overline{z}\rangle \tag{2.22}$$

and unitary, as it maps one-to-one the complete orthonormal set (2.8) onto the complete orthonormal set (2.11). Thus equations (2.21) and (2.22) describe, via coherent states, the isometry between the abstract Fock space $\mathcal{F}$ and its realization $F^2(\mathbb{C}, d\mu)$.

It is worth noticing that the concept of coherent states is far more general than presented here. Coherent states [10] can be defined in any Hilbert space providing a unitary irreducible representation for any compact semi-simple Lie algebra. In those cases, generalized coherent states are labeled by points in arising complex manifolds; vectors in the Hilbert space are realized as analytic functions on these manifolds.

2.4. Isomorphism between the Fock Space and $l^2(\mathbb{C})$

A vector $|\psi\rangle \in \mathcal{F}$ can be expanded in the orthonormal set (2.11) as $|\psi\rangle = \sum_m \psi|m\rangle$ with coefficients $\psi_m := \langle m|\psi\rangle$ in $l^2(\mathbb{C})$, the Hilbert space of complex square summable sequences. The linear transformation $V : \mathcal{F} \to l^2(\mathbb{C})$ defined by

$$V(|\psi\rangle) = \{\langle m|\psi\rangle\}_{m\in\mathbb{N}} \tag{2.23}$$

is unitary, preserving the inner product

$$\sum_m \eta_m^* \psi_m = \sum_m \langle \eta|m\rangle \langle m|\psi\rangle = \langle \eta|\psi\rangle \tag{2.24}$$

because of (2.9). This makes $l^2(\mathbb{C})$ a realization the Fock space $\mathcal{F}$, mapping the set $\{|n\rangle\}$ in (2.11) one-to one with the canonical complete orhonormal set of sequences $\{\langle m|n\rangle\}_{m\in\mathbb{N}}$.

Given the unitary isomorphism V, a linear operator A in $\mathcal{F}$ is realized in $l^2(\mathbb{C})$ as the unitarily transformed operator $t = VAV^{-1}$. In this way, the creation and annihilation operators are realized in $l^2(\mathbb{C})$. Explicitly, if a vector $|\psi\rangle$ belongs to the domain of A, t acts on the realization $\{\psi_m\}$ as $t(\{\psi_m\}) = \{\psi'_m\}$ with

$$\psi'_m = \langle m|A|\psi\rangle \tag{2.25}$$

Assuming that all the canonical basis vectors are in the domain of A on can use (2.9) again to get

$$\psi'_m = \sum_n \langle m|A|n\rangle \psi_n, \tag{2.26}$$

expressing as convergent series the elements of $\{\psi'_m\} \in l^2(\mathbb{C})$. The coefficients $\langle m|A|n\rangle$ are known in quantum physics as matrix elements of the operator A.

All the above can be used as a way to establish the isometry between $F^2(\mathbb{C}, d\mu)$ and $l^2(\mathbb{C})$, as realizations of the same Fock space [2] $\mathcal{F}$. The isometry is given transitively by $W : F^2(\mathbb{C}, d\mu) \to l^2(\mathbb{C})$ defined by

$$W = VU^{-1}. \tag{2.27}$$

3. Toeplitz Operators as Realization of Anti-Wick Operators

In this section we show that some linear operators in the Fock space $\mathcal{F}$ are realized as Toeplitz operators in $F^2(\mathbb{C}, d\mu)$. This in turn provides an abstract structure for their analysis.

We first introduce a class of integral operators on $\mathcal{F}$ formally given by a diagonal expression in the coherent states set:

Definition 3. Given a measurable function $\varphi(w, \overline{w})$, not necessarily analytic, let

$$A_\varphi = \int_{\mathbb{C}} \frac{dw\, d\overline{w}}{\pi}\, |\overline{w}\rangle \varphi(w, \overline{w}) \langle \overline{w}| \tag{3.1}$$

The function $\varphi(w, \overline{w})$ is known as the *anti-Wick* or contravariant symbol of A_φ [1], and we refer to A_φ as an anti-Wick operator. The integral in (3.1) is understood in the weak sense, meaning here that for any vector $|\psi\rangle$ in the domain of A_φ and for any vector $|\eta\rangle$ in $F^2(\mathbb{C}, d\mu)$,

$$\langle \eta|A_\varphi|\psi\rangle = \int_{\mathbb{C}} \frac{dw\, d\overline{w}}{\pi}\, \langle \eta|\overline{w}\rangle \varphi(w, \overline{w}) \langle \overline{w}|\psi\rangle. \tag{3.2}$$

We emphasize that, for a given symbol $\varphi(w, \overline{w})$, the operator in (3.1) may be not well defined in $\mathcal{F}$. According to original works by Berezin[1], we present the following

[2] The present construction is one of the many ways to show the isometry between the Segal-Bargmann space $F^2(\mathbb{C}, d\mu)$ and $l^2(\mathbb{C})$. Indeed, since the Fock space is a unitary irreducible representation of the Heisenberg-Weyl group [10], these isomorphisms are just examples of the celebrated theorem by Stone and von Neumann [11] stating that any two unitary irreducible representations of this group are unitarily equivalent.

Definition 4. Given a measurable function $\varphi(z,\overline{z})$, the *natural domain* of the anti-Wick operator A_φ is the subset of $\mathcal{F}$

$$Dom(A_\varphi) = \{|\psi\rangle \in \mathcal{F} : \varphi\psi \in L^2(\mathbb{C}, d\mu)\}, \tag{3.3}$$

where $\psi(z) = e^{\frac{1}{2}|z|^2}\langle \overline{z}|\psi\rangle$.

Of course, this is the subset of vectors $|\psi\rangle \in \mathcal{F}$ with $F^2(\mathbb{C}, d\mu)$ realization $\psi(z)$ in the natural domain (2.5) of the Toeplitz operator T_φ. For such vectors, (3.2) provides a strong sense to the formal definition of the anti-Wick operator in (3.1). The operator A_φ is *a well defined* endomorphism if and only if its natural domain is dense in $\mathcal{F}$.

Next, we explore the realization of anti-Wick operators in $F^2(\mathbb{C}, d\mu)$. By realization of an operator A in $\mathcal{F}$ as an operator in $F^2(\mathbb{C}, d\mu)$, we mean the unitarily transformed operator $T = UAU^{-1}$, where U is the isometry in (2.21). We have the following relation:

Proposition 5. Given a measurable function $\varphi(z,\overline{z})$ defined on $\mathbb{C}$, not necessarily analytic, the operator $UA_\varphi U^{-1}$ with U the isometry in (2.21), realizing the anti-Wick operator A_φ in $F^2(\mathbb{C}, d\mu)$, is the Toeplitz operator T_φ.

Proof. Let $|\psi\rangle \in Dom(A_\varphi)$ and denote $\psi(z) = e^{\frac{1}{2}|z|^2}\langle \overline{z}|\psi\rangle$ its realization in $F^2(\mathbb{C}, d\mu)$. The realization of A_φ in $F^2(\mathbb{C}, d\mu)$ acts on $\psi(z)$ as

$$UA_\varphi U^{-1}(\psi)(z) = UA_\varphi|\psi\rangle = e^{\frac{1}{2}|z|^2}\langle \overline{z}|A_\varphi|\psi\rangle. \tag{3.4}$$

The definition of A_φ in (3.1) leads to

$$UA_\varphi U^{-1}(\psi)(z) = \int_{\mathbb{C}} d\mu(w) e^{z\overline{w}} \varphi(w,\overline{w})\psi(w), \tag{3.5}$$

a well defined Bargmann projection in $L^2(\mathbb{C}, d\mu)$,

$$UA_\varphi U^{-1}(\psi)(z) = P(\varphi\psi)(z). \tag{3.6}$$

This is just the operator T_φ as defined in (2.3). The natural domain A_φ defined in (3.3) is in one-to-one correspondence with the natural domain of T_φ defined in (2.5). □

This result enables the discussion of Toeplitz operators properties in terms of the more abstract anti-Wick operators. In particular, one can explore the way that Toeplitz operators are realized in other isometric Fock space realizations.

As mentioned in the Introduction, our interest points towards realizations on complex sequences.

To end this section, let us comment on the origin of the anti-Wick name. An operator form like (3.1) can be obtained for any operator with a polynomial expression in $\widehat{a}$ and $\widehat{a}^\dagger$ [12]. The corresponding symbol is a polynomial in w and $\overline{w}$ obtained by replacing $\widehat{a} \to w$, $\widehat{a}^\dagger \to \overline{w}$ after the operator terms are arranged (by means of commutations) with creation operators acting before annihilation ones. This order is opposite to the normal or Wick order in quantum physics, where annihilation operators act before creation ones, hence it is called anti-Wick order. A well known example of operator with anti-Wick form is the (dimensionless) Hamiltonian operator for a harmonic oscillator, usually written in normal order as $\widehat{H} = \widehat{a}^\dagger\widehat{a} + 1/2$. Its anti-Wick symbol results $H(w,\overline{w}) = w\overline{w} - 1/2$. As we mentioned above, coherent states are defined in more general contexts than the present one, leading to analytic functions on complex manifolds. Contravariant operators, as defined in (3.1), lead to generalized Toeplitz operators on those manifolds.[13]

4. Toeplitz Operators with Radial Symbols

Toeplitz operators with radial symbols $\varphi(z,\overline{z}) = \varphi(|z|)$ are relevant in physics. They arise as realizations of anti-Wick operators with a detailed balance of $\widehat{a}^\dagger$ and $\widehat{a}$ operations (called *particle number conservation*); for operators with polynomial form in $\widehat{a}^\dagger$ and $\widehat{a}$, a radial symbol means that each term must contain the same number of $\widehat{a}^\dagger$ and $\widehat{a}$ factors, as in the harmonic oscillator example.

The main point in our discussion is that well defined Toeplitz operators with radial symbols are particularly simple when realized in $l^2(\mathbb{C})$. This property has been worked out in detail in Ref. [5], as a tool to analyze some Toeplitz operators properties. However, not any radial symbol gives rise to a well defined Toeplitz operator. This simple fact has been source of erroneous assertions in [5] and following papers. The aim of the present work is to give a correct formulation of these assertions.

The first step is the definition of an appropriate class of radial symbols:

Definition 6. We denote $\mathcal{P}$ the class of radial symbols

$$\mathcal{P} = \{\varphi(|z|) : \forall n \in \mathbb{N},\, z^n \in Dom(T_\varphi)\}\,. \tag{4.1}$$

where $Dom(T_\varphi)$, defined in (2.5), is the natural domain of the Toeplitz operator T_φ.

For symbols in this class, a Toeplitz operator T_φ is well defined, having a dense domain in $F^2(\mathbb{C}, d\mu)$ containing at least any polynomial (correspondingly, the anti-Wick operator A_φ has a dense domain in $\mathcal{F}$, containing any vector $|n\rangle$). We are in conditions to prove:

Theorem 7. *Given a radial symbol $\varphi(|z|) \in \mathcal{P}$, the Toeplitz operator T_φ defined in (2.3) is unitarily transformed by the isometry W defined in (2.27) into a linear operator which is diagonal in the canonical basis of $l^2(\mathbb{C})$.*

Proof. Given a radial symbol $\varphi(|z|) \in \mathcal{P}$, consider the anti-Wick operator A_φ defined in (3.1). Its natural domain, in one-to-one correspondence with the natural domain of the Toeplitz operator T_φ, contains any vector $|n\rangle$ in the canonical basis (2.8), so that the matrix elements $\langle m|A_\varphi|n\rangle$ are well defined. They are given by

$$\langle m|A_\varphi|n\rangle = \int_{\mathbb{C}} d\mu(z)\varphi(|z|)\frac{\overline{z}^m}{\sqrt{m!}}\frac{z^n}{\sqrt{n!}}. \tag{4.2}$$

Here $\varphi(|z|) \in \mathcal{P}$ ensures that the integral is a well defined inner product in $L^2(\mathbb{C}, d\mu)$. Using polar coordinates for z it is straightforward to show that

$$\langle m|A_\varphi|n\rangle = \delta_{mn}\varphi_n \tag{4.3}$$

with δ_{mn} the Krnecker delta and

$$\varphi_n = \frac{2}{n!}\int_0^\infty \varphi(r)r^{2n+1}e^{-r^2}dr. \tag{4.4}$$

Then, the linear operator $t_\varphi = VA_\varphi V^{-1}$ realizing A_φ in $l^2(\mathbb{C})$, expressed in the form (2.26), has only diagonal non vanishing matrix elements. Applied to the sequence $\{\psi_n\}$, it gives

$$t_\varphi(\{\psi_n\}) = \{\varphi_n\psi_n\}.$$

On the other hand, from Proposition 5, A_φ is unitarily transformed by U in (2.21) to the Toeplitz operator T_φ with symbol $\varphi(|z|)$, $T_\varphi = UA_\varphi U^{-1}$. Then $WT_\varphi W^{-1} = t_\varphi$. □

A key point in the proof above is that A_φ can be applied to any vector in the set (2.8). Without this assumption, some of the matrix elements in (4.2) would be not well defined and the unitary equivalence between T_φ and t_φ would fail. This is the content of

Proposition 8. Given a radial symbol $\varphi(|z|) \notin \mathcal{P}$, the Toeplitz operator T_φ is not unitarily equivalent to any diagonal linear operator in the canonical basis of $l^2(\mathbb{C})$.

Proof. Let $\varphi(|z|) \notin \mathcal{P}$, then there exists $n \in \mathbb{N}$ such that $n(z) = \frac{z^n}{\sqrt{n!}}$ does not belong to the natural domain of the Toeplitz operator T_φ. Now assume that an operator $S : F^2(\mathbb{C}, d\mu) \to l^2(\mathbb{C})$ is a unitary transformation. Then $ST_\varphi S^{-1}$ is not defined for the finite sequence $S(n(z))$. But finite sequences belong to the domain of any diagonal linear operator in $l^2(\mathbb{C})$. Thus, $ST_\varphi S^{-1}$ cannot be a diagonal operator in the canonical basis of $l^2(\mathbb{C})$. □

From Theorem 7 one learns that well defined Toeplitz operators with radial symbols are indeed very simple operators. The unitarily equivalent diagonal form in $l^2(\mathbb{C})$ allows to read the infinite numerable spectrum with eigenvalues φ_n. Using the isometry W, the corresponding normalized eigenfunctions in $F^2(\mathbb{C}, d\mu)$ are $z^n/\sqrt{n!}$. All of this allows to characterize properties such as boundness or compactness of a radial Toeplitz operator in terms of its symbol.

Remark. *The importance of caring about the domain of definition of the operators.* A study of radial Toeplitz operators has been carried out in Ref. [5], with some erroneous statements. The authors use the well established isometry between $F^2(\mathbb{C}, d\mu)$ and $l^2(\mathbb{C})$, but they consider a class of radial symbols $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, defined as the set of all measurable functions $\varphi(r)$ on $\mathbb{R}_+$ such that

$$\forall m \in \mathbb{N}, \quad \int_0^\infty |\varphi(r)| r^m e^{-r^2} dr < \infty \,, \tag{4.5}$$

and claim that Toeplitz operators with radial symbols in this class are unitarily equivalent to diagonal operators in $l^2(\mathbb{C})$. We observe that the condition (4.5) just makes φ_n in (4.4) computable. As we showed in Proposition 8, $\mathcal{P}$ is the largest class of radial symbols such that Toeplitz operators can unitarily transformed into diagonal operators in $l^2(\mathbb{C})$. The class $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ is too wide and contains symbols that are not in our class $\mathcal{P}$. To make this clear we present the following

Example 9. Let $\varphi(r) = e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)r^2}$. It belongs to $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ allowing to compute, by (4.4), an associated (bounded) sequence $\varphi_n = (\frac{1}{2} - \frac{\sqrt{3}}{2}i)^{-(n+1)}$. However, $\varphi(|w|) \notin \mathcal{P}$: it is enough to observe that for $n = 0$, $n(z)$ does not belong to $Dom(T_\varphi)$. In consequence, the Toeplitz operator T_φ is not unitarily equivalent any diagonal operator in $l^2(\mathbb{C})$. Moreover, $Dom(T_\varphi) = \{0\}$ in this example, while $Dom(t_\varphi) = l^2(\mathbb{C})$!

Then, discussion of properties such as boundness or compactness of radial Toeplitz operators, founded on the mapping in Theorem 7, must be restricted to operators with symbols in $\mathcal{P}$.

5. Related Open Problems

We have proven that $\mathcal{P}$ is the largest class of radial symbols where Toeplitz operators can be unitarily mapped into diagonal operators in $l^2(\mathbb{C})$. The inverse problem is then relevant: given a sequence defining the spectrum of a diagonal operator in the canonical basis of $l^2(\mathbb{C})$, is it possible to map such operator into a Toeplitz operator in $F^2(\mathbb{C}, d\mu)$?

In Ref. [5], by means of a nice construction based on analytic continuation and Fourier transforms, the authors were able to build functions in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$ producing by (4.4) any *bounded* complex sequence. However, as discussed in Section 4, taking a function in this set as a radial symbol for a Toeplitz operator does not guarantee that it is equivalent with a diagonal operator in $l^2(\mathbb{C})$. In consequence, their claim in Theorem 3.7 mapping diagonal operators in $l^2(\mathbb{C})$ with bounded spectrum into Toeplitz operators with radial symbols is not valid. In particular, for the sequence in Example 9, the resulting symbol leads to a Toeplitz operator with trivial domain.

The described inverse problem is still an open one. We limit ourselves here to provide a family of situations (generalizing the previous example) that illustrate several possibilities.

Example 10. Consider the sequences $\{\gamma_n^{(k)}\}$ with $\gamma_n^{(k)} = k^{-n}$, $k \in \mathbb{C}$.

For $Re(k) > 1/2$, these sequences are obtained by (4.4) from a radial symbols $\gamma^{(k)}(|z|) = ke^{(1-k)|z|^2}$ belonging to $\mathcal{P}$. Then, The Toeplitz operator $T_\gamma^{(k)}$ is equivalent to the operator in $l^2(\mathbb{C})$ with diagonal matrix elements $\{\gamma_n^{(k)}\}$. This holds even for unbounded sequences, when $|k| < 1$.

For $0 < Re(k) \le 1/2$, these sequences are still obtained by (4.4) from a radial symbols $\gamma^{(k)}(|z|) = ke^{(1-k)|z|^2}$ belonging to $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, but not to $\mathcal{P}$. The Example 9 falls in this case, with $|k| = 1$.

We make some remarks on these examples:

— Symbols $\gamma^{(k)}(|z|) = ke^{(1-k)|z|^2}$ belong simultaneously to the sets $\mathcal{P}$ in (4.1), Folland's class in (2.6) and Coburn's class in (2.7), or to none of them. They thus illustrate the equivalence of Toeplitz radial operators with diagonal operators in $l^2(\mathbb{C})$ also in those contexts.

— As an application in quantum physics, the *density matrix operator* $\widehat{\rho}$ at inverse temperature β is constructed from a Hamiltonian $\widehat{H}$ as proportional to $e^{-\beta\widehat{H}}$. For a harmonic oscillator with $\widehat{H} = \widehat{a}^\dagger\widehat{a}+1/2$, the operator $\widehat{\rho}$ can be written as an anti-Wick operator with a symbol of the present form, proportional to $e^{(1-k)|z|^2}$ with $k > 1$. The spectrum in this case is bounded.

— Finally, we recall the kind of example provided in Ref. [4] as an obstruction for the composition of Toeplitz operators to be closed, which is included in our analysis. Indeed, consider a symbol $\gamma^{(k)}(|z|)$ with $Re(k) > 1/2$ but $0 < Re(k^2) < 1/2$, and $|k| = 1$ (such as $k = \frac{3}{4} + \frac{\sqrt{7}}{4}$). The Toeplitz operator $T_\gamma^{(k)}$ is equivalent to a diagonal operator in $l^2(\mathbb{C})$ with matrix elements $1/k^n$. Now consider the composition of $T_\gamma^{(k)}$ with itself: it is straightforwardly worked out when realized in $l^2(\mathbb{C})$, as a diagonal operator with matrix elements $1/k^{2n}$. Following [5], one could construct from this bounded sequence a symbol $\eta(|z|) = k^2 e^{(1-k^2)|z|^2}$ in $L_1^\infty(\mathbb{R}_+, e^{-r^2})$, but not belonging to $\mathcal{P}$. While their claim in Theorem 3.7 implies that $T_\gamma^{(k)} T_\gamma^{(k)}$ can be written as a Toeplitz operator with symbol $\eta(|z|)$, our results prove, as discussed in Ref. [4], that $T_\varphi T_\varphi$ is not a well defined Toeplitz operator, *i.e.* the composition is not closed.

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